A multi-level fast-marching method for the minimum time problem

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We introduce a new numerical method to approximate the solutions of a class of stationary Hamilton-Jacobi partial differential equations arising from minimum time optimal control problems. We rely on several grid approximations, and look for the optimal trajectories by using the coarse grid approximations to reduce the search space in fine grids. This may be thought of as an infinitesimal version of the "highway hierarchy" method which has been developed to solve shortest path problems (with discrete time and discrete state), see [2]. Using fastmarching method [3], we obtain, for each level, an approximate value function on a sub-domain of the state space.

We prove that, by using this multi-level algorithm, the final approximation error is as good as the one obtained by discretizing directly the whole domain with the finest grid. Then, the sequences of approximate values and optimal trajectories obtained in this way do converge to the value and optimal trajectory of the minimum time optimal control problem.

We also show that the number of arithmetic operations and the size of the memory needed to get an error of ε are considerably reduced. Recall that the number of arithmetic operations of conventional grid-based methods is in the order of $\widetilde{O}(K_d \varepsilon^{-\frac{d}{\gamma}})$, where d is the dimension of the problem, $0 < \gamma \leq 1$ is the convergence rate of the classical numerical scheme, $K_d \in [2d, L^d]$ for some constant L depending on the diameter of discrete neighborhoods, and $\widetilde{O}$ ignores the logarithmic factors. For our multi-level fastmarching method, with suitable parameters, the number of arithmetic operations is in the order of $\widetilde{O}(C^d \varepsilon^{-\frac{1+(d-1)(1-\gamma\beta)}{\gamma}})$, where $C > 1$ is a constant depending on the problem characteristics, and $0 < \beta \leq 1$ measures the "stiffness" of the value function around optimal trajectories. When $\gamma = 1$ (see for instance [1]), and the stiffness is high ($\beta = 1$), this complexity becomes in $\widetilde{O}(C^d \varepsilon^{-1})$.

We illustrate the approach by solving Hamilton-Jacobi equations of eikonal type up to dimension 6.

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[2] P. Sanders, D. Schultes. *Engineering highway hierarchies.* Journal of Experimental Algorithmics (JEA), **17**, 1–1, 2012.

[3] J. A. Sethian. *A fast marching level set method for monotonically advancing fronts.* Proceedings of the National Academy of Sciences, **93(4)**, 1591–1595, 1996.

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