Optimal control of fractional Sturm-Liouville wave equations on a star graph

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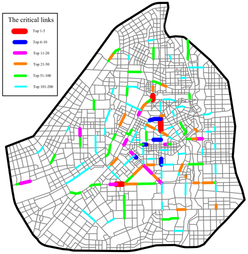
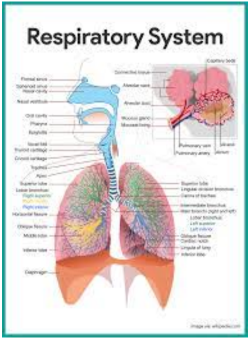
1. Introduction and motivation
   - Network
   - Sturm-Liouville Theory
   - Fractional Calculus
   - Spectral Theorem

2. Fractional Sturm–Liouville wave equations

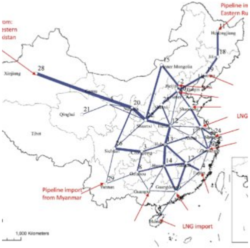
3. Existence of minimizers and optimality conditions



Mynd: Traffic flow network



Mynd: Respiratory system network



Mynd: Gas-pipeline network of China

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3. Existence of minimizers and optimality conditions

A Sturm-Liouville problem is a problem generated by the following equation,

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+l(x)y=\lambda r(x)y\,.$$

Importance of Sturm-Liouville:

- All second order equation can be transform into a Sturm-Liouville problem
- Under suitable initial and boundary condition, the Sturm-Liouville operator is self-adjoint.

The fractional calculus theory

- generalizes the classical differential calculus
- takes into account the non-locality and memory effect
- models physical and engineering processes that are found to be best described by fractional differential equations (anomalous diffusion, viscoelastic models, aeroelastic, etc...)

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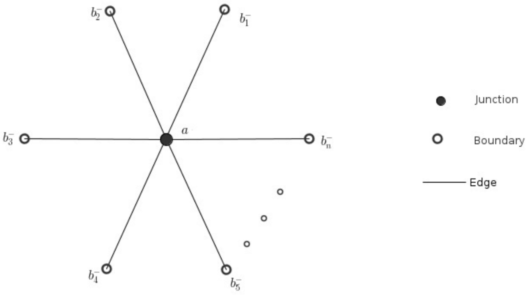
2 Fractional Sturm–Liouville wave equations

3 Existence of minimizers and optimality conditions

we consider a network in the form of a star graph



Mynd: Star graph with n edges

We define the spaces

$$H^{\alpha}_{a^+}(a,b_i) = AC^{\alpha,2}_{a^+} \cap L^2(a,b_i)$$

$$\mathbb{H}^{\alpha}_{a} := \prod_{i=1}^{n} H^{\alpha}_{a^+}(a,b_i) \qquad \mathbb{L}^2 = \prod_{i=1}^{n} L^2(a,b_i).$$

with the norms

$$\|\rho\|^2_{\mathbb{L}^2} = \sum_{i=1}^{n} \|\rho^i\|^2_{L^2(a,b_i)}, \quad \rho = (\rho^i)_i \in \mathbb{L}^2,$$

$$\|\rho\|^2_{\mathbb{H}^{\alpha}_a} = \sum_{i=1}^{n} \|\rho^i\|^2_{H^{\alpha}_{a^+}(a,b_i)} := \sum_{i=1}^{n} \left(\|\rho^i\|^2_{L^2(a,b_i)} + \|\mathbb{D}^{\alpha}_{a^+}\rho^i\|^2_{L^2(a,b_i)} \right).$$

We define the space $\mathcal{V}_i$ by

$$\mathcal{V}_i := \left\{ \rho^i \in H^{\alpha}_{a^+}(a, b_i) : \mathcal{D}^{\alpha}_{b^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} \rho^i) \in H^{1-\alpha}_{b^-}(a, b_i) \right\}$$

$$\mathbb{V} = \left\{ (\rho^i)_i \in \prod_{i=1}^{n} \mathcal{V}_i : I^{1-\alpha}_{a^+} \rho^i(a^+, \cdot) = I^{1-\alpha}_{a^+} \rho^j(a^+, \cdot), i \neq j, \ i, j = 1, \dots, n \right.$$
$$\left. \sum_{i=1}^{n} \beta^i(a) \mathbb{D}^{\alpha}_{a^+} \rho^i(a^+, \cdot) = 0 \right\}$$

with the norm $\|\rho\|_{\mathbb{V}} = \|\rho\|_{\mathbb{H}^{\alpha}_a} \quad \forall \rho = (\rho^i) \in \mathbb{V}$.

For any $\rho, \phi \in \mathbb{V}$, we define the bilinear form $\mathbb{E}(\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ by:

$$\mathbb{E}(\rho,\phi) = \sum_{i=1}^{N} \int_{Q_i} (\beta^i(x)\mathbb{D}^{\alpha}_{a^+}\phi^i)(x,t)\mathbb{D}^{\alpha}_{a^+}(\rho^i))(x,t)dxdt$$
$$+ \sum_{i=1}^{N} \int_{Q_i} q^i(x)\phi^i(x,t)\rho^i(x,t)dx\,dt,$$

Assumption

$\beta^i \in \mathcal{C}^1([a,b_i])$, $i = 1,\cdots,n$ and $q^i \in \mathcal{C}([a,b_i])$, $i = 1,\cdots,n$ such that

$$\underline{q^0} := \min_{1\le i\le n} q^i, \quad \underline{\beta^0} := \min_{1\le i\le n} \beta^i,$$
$$\overline{q} := \max_{1\le i\le n} \|q^i\|_\infty, \quad \overline{\beta} := \max_{1\le i\le n} \|\beta^i\|_\infty$$

Then, we define the self adjoint operator $\mathcal{A}$ as follows: For

We define the operator $\mathcal{A}$ as follows: For $\rho = (\rho^i)_i$, we let

$$\mathcal{A}\rho = ((\mathcal{A}\rho)^i)_i = \left(\mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+}\rho^i) + q^i\rho^i\right)_i, \quad i = 1, \dots, n. \quad (4)$$

with

$$D(\mathcal{A}) : \{(\rho^i)_i \in \mathbb{V}; I^{1-\alpha}_{a^+}\rho^i(b_i^-) = 0, \; i = 1, \dots, m, \;\; \beta^i(b_i)\mathbb{D}^{\alpha}_{a^+}\rho^i(b_i^-) = 0, i = m+1, \cdots$$

and $\|\phi\|^2_{D(\mathcal{A})} := \|\phi\|^2_{\mathbb{V}}$.

- For $1/2 < \alpha < 1$, the embedding $\mathbb{H}_a^\alpha \hookrightarrow \mathbb{L}^2$ is compact.
- $\mathcal{A}$ is a self-adjoint operator.
- $\mathbb{E}(\cdot, \cdot)$ is continuous and coercive on $\mathbb{V}$.

Conclusion: there exists an orthonormal basis of $\mathbb{L}^2$, $(\varphi_n)_n$ consisting of eigenvectors of $\mathcal{A}$, moreover from the coercivity and the nonnegativity of $\mathbb{E}(\cdot, \cdot)$, the eigenvalues associated to $(\varphi_n)_n$ verifies

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \quad \text{with} \quad \lim_{n \to \infty} \lambda_n = +\infty.$$

Problem setting

$$\min_{v\in\mathcal{U}_{ad}} \sum_{i=1}^{n} \left(\frac{1}{2}\int_a^{b_i} \left|y^i(T) - z_d^{0,i}\right|^2 \mathrm{d}x + \frac{1}{2}\int_a^{b_i} \left|y_t^i(T) - z_d^{1,i}\right|^2 \mathrm{d}x + \frac{N}{2}\int_a^{b_i} |v_i|^2 \mathrm{d}x \right),$$

where $z_d^0 = (z_d^{0,i})_i \in \mathbb{V}$, $z_d^1 = (z_d^{1,i})_i \in \mathbb{L}^2$ and $y = (y^i)_i$ satisfies the following wave equation:

$$\begin{cases} y_{tt}^i + \mathcal{D}_{b_i-}^{\alpha}(\beta^i \mathbb{D}_{a+}^{\alpha} y^i) + q^i y^i = f^i, & \text{in } (a,b_i)\times(0,T),\ i=1,\dots,n,\\ I_{a+}^{1-\alpha} y^i(a^+) = I_{a+}^{1-\alpha} y^j(a^+) & \text{in } (0,T),\ i\neq j = 1,\dots,n,\\ \sum_{i=1}^{n} \beta^i(a)\mathbb{D}_{a+}^{\alpha} y^i(a^+) = 0, & \text{in } (0,T),\\ I_{a+}^{1-\alpha} y^1(b_1^-) = 0, & \text{in } (0,T),\\ I_{a+}^{1-\alpha} y^i(b_i^-) = h_i, & \text{in } (0,T),\ i=2,\dots,m,\\ \beta^i(b_i)\mathbb{D}_{a+}^{\alpha} y^i(b_i^-) = g_i, & \text{in } (0,T),\ i=m+1,\dots,n,\\ y^i(0) = y^{0,i}, & \text{in } (a,b_i),\ i=1,\dots,n,\\ y_t^i(0) = v_i, & \text{in } (a,b_i),\ i=1,\dots,n. \end{cases} \quad (5)$$

We consider the following:

$$\begin{cases} y^i_{tt} + \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} y^i) + q^i y^i = f^i & \text{in} \quad Q_i,\ i = 1, \dots, n, \\ I^{1-\alpha}_{a^+} y^i(a^+, \cdot) = I^{1-\alpha}_{a^+} y^j(a^+, \cdot) & \text{in} \quad (0, T),\ i \neq j = 1, \dots, n, \\ \sum_{i=1}^{n} \beta^i(a) \mathbb{D}^{\alpha}_{a^+} y^i(a^+, \cdot) = 0 & \text{in} \quad (0, T), \\ I^{1-\alpha}_{a^+} y^i(b_i^-, \cdot) = 0 & \text{in} \quad (0, T),\ i = 1, \dots, m \\ \beta^i(b_i) \mathbb{D}^{\alpha}_{a^+} y^i(b_i^-, \cdot) = 0 & \text{in} \quad (0, T),\ i = m+1, \dots, n, \\ y^i(0) = y^{0,i} & \text{in} \quad (a, b_i),\ \ i = 1, \dots, n, \\ y^i_t(0) = v_i & \text{in} \quad (a, b_i),\ \ i = 1, \dots, n, \end{cases} \tag{6}$$

where $f = (f^i) \in L^2((0,T); \mathbb{V}^\star)$, $y^0 = (y^{0,i}) \in D(\mathcal{A})$, $v = (v_i) \in \mathbb{L}^2$.

The Cauchy problem (6) becomes

$$\begin{cases} y_{tt} + \mathcal{A}y = f(t) & \text{for} \quad t \in (0, T), \\ y(\cdot, 0) = y^0, \\ y_t(\cdot, 0) = v, \end{cases} \tag{7}$$

Definition

A function $y = (y^i)$ is said to be a weak solution of (6) in $(0, T)$, $T > 0$, if the following assertions hold.

- Regularity: $y \in C^1([0, T]; \mathbb{L}^2) \cap C([0, T]; D(\mathcal{A}))$
- Initial condition: $y^i(\cdot, 0) = y^{i,0}$ in (a, b_i), $i = 1, \cdots, n$; $y^i_t(\cdot, 0) = v_i$ in (a, b_i), $i = 1, \cdots, n$
- Variational identity: $\langle y_{tt}, \zeta\rangle_{\mathbb{V}^\star, \mathbb{V}} + \mathbb{E}(y(t, \cdot), \zeta) = \langle f(t, \cdot), \varphi\rangle_{\mathbb{V}^\star, \mathbb{V}}$ for every $\varphi \in D(\mathcal{A})$, and a.e. $t \in (0, T)$.

Theorem

Let $1/2 < \alpha < 1$, $f \in L^2((0,T);\mathbb{V}^\star)$ *and* $y^0 \in D(\mathcal{A})$, $v \in \mathbb{L}^2$, *the system* (6) *has a unique weak solution* $y = (y^i)$ *given by*

$$y(t,\cdot) = \sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{\mu_n}} \langle v, \zeta_n \rangle_{\mathbb{L}^2} \sin(\sqrt{\mu_n} t) + \langle y^0, \zeta_n \rangle_{\mathbb{L}^2} \cos(\sqrt{\mu_n} t) + \frac{1}{\sqrt{\mu_n}} \int_0^t \sin(\sqrt{\mu_n}(t-s)) \langle f(s), \zeta_n \rangle_{\mathbb{V}^\star,\mathbb{V}} \, \mathrm{d}s \right\} \zeta_n. \tag{8}$$

Moreover,

$$\frac{1}{2}\|y_t\|^2_{\mathcal{C}([0,T];\mathbb{L}^2)} \leq 2T\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \max\left(\overline{\beta}, \overline{q}\right) \|y^0\|^2_{\mathbb{H}^\alpha_a} \tag{9a}$$

$$\|y\|^2_{\mathcal{C}([0,T];\mathbb{H}^\alpha_a))} \leq \frac{2}{\min\left(\underline{\beta^0}, \underline{q^0}\right)} \left(2T\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \max\left(\overline{\beta}, \overline{q}\right) \|y^0\|^2_{\mathbb{H}^\alpha_a}\right) \tag{9b}$$

steps of the prove

- Multiply equation (7) by φ_n to obtain

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} y_n(t) + \lambda_n y_n(t) &= f_n(t) \quad \text{for} \quad t \in (0,T), \\ y_n(0) &= y_n^0, \\ y_n'(0) &= v_n, \end{cases} \tag{10}$$

where
$y_n(t) = (y(t), \varphi_n)_{\mathbb{L}^2}$, $f_n(t) = \langle f(t), \varphi_n \rangle_{\mathbb{V}^\star, \mathbb{V}}$, $y_n^0 = (y^0, \varphi_n)_{\mathbb{L}^2}$,
and $v_n = (v, \varphi_n)_{\mathbb{L}^2}$

- Take the Laplace transform of the equation (10).
- Using the Laplace transforms of cos, sin and convolution functions, recover y_n. then write $y(t) = \sum_{n=1}^{\infty} y_n(t) \varphi_n$.

In what follows, we shall transform that problem into an homogeneous type problem.

Let $(g_i)_{i=2\cdots m}$ and $(h_i)_{i=m+1\cdots n} \in H^2(0,T)$ be such that $h_i(0) = g_i(0) = 0,\ i = 1, \cdots, n$, we set

$$\|\overline{g}\|_{\mathcal{C}([0,T])} = \max_{2\leq i\leq m} \|g_i\|_{\mathcal{C}([0,T])}, \quad \|\overline{h}\|_{\mathcal{C}([0,T])} = \max_{m+1\leq i\leq n} \|h_i\|_{\mathcal{C}([0,T])} \tag{11a}$$

$$\|\overline{g_t}\|_{\mathcal{C}([0,T])} = \max_{2\leq i\leq m} \|g_i'\|_{\mathcal{C}([0,T])}, \quad \|\overline{h}_t\|_{\mathcal{C}([0,T])} = \max_{m+1\leq i\leq n} \|h_i'\|_{\mathcal{C}([0,T])} \tag{11b}$$

We consider the function $w = (w^i)_{i=1,\cdots,n}$, where w^i are giving by :

$$w^i(x,t) = \begin{cases} 0 & i = 1 \\ \dfrac{2g_i(t)}{\Gamma(\alpha+2)(b_i-a)^2}(x-a)^{\alpha+1} & i = 2, \cdots, m \\ \dfrac{h_i(t)}{\Gamma(\alpha+2)\beta^i(b_i)(b_i-a)}(x-a)^{\alpha+1} & i = m+1, \cdots, n. \end{cases} \tag{12}$$

Lemma

Assume that Assumption 1 holds. Let $(g_i)_{i=2\cdots m} \in \left(H^2(0,T)\right)^{m-1}$ *and* $(h_i)_{i=m+1\cdots n} \in \left(H^2(0,T)\right)^{n-m}$ *be such that* (11) *holds. Let also* $w = (w^i)$ *be defined as in* (12)*. Then we have that* $w^i, \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} w^i) \in L^2(Q_i)$, $w^i_t \in \mathcal{C}([0,T]; L^2(a,b_i))$ *and* $w^i_{tt} \in L^2(Q_i)$.
Moreover,

$$\sup_{t\in[0,T]} \|w(t)\|^2_{\mathbb{H}^{\alpha}_a} \le C(\alpha, \underline{b}, \overline{b}, \underline{\beta^0}, n)\left(\|\overline{g}\|^2_{\mathcal{C}([0,T])} + \|\overline{h}\|^2_{\mathcal{C}([0,T])}\right), \tag{13a}$$

$$\|w_t(0)\|^2_{\mathbb{L}^2} \le C(\alpha, \overline{b}, \underline{\beta^0}, n)\left(\|\overline{g}_t\|^2_{\mathcal{C}([0,T])} + \|\overline{h}_t\|^2_{\mathcal{C}([0,T])}\right), \tag{13b}$$

where $\overline{b} = \max_{1\le i\le n}(b_i)$ *and* $\underline{b} = \min_{1\le i\le n}(b_i)$.

and setting $z = (z^i)_{i=1,\cdots,n}$ where $z^i(x,t) = y^i(x,t) - w^i(x,t)$, it follows that z satisfies the homogeneous system

$$\left\{\begin{array}{lcllll}
z^i_{tt} + \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} z^i) + q^i z^i & = & \tilde{f}^i & \text{in} & Q_i, \ i = 1, \ldots, n, \\
I^{1-\alpha}_{a^+} z^i(a^+, \cdot) & = & I^{1-\alpha}_{a^+} z^j(a^+, \cdot) & \text{in} & (0,T), \ i \neq j = 1, \ldots, n \\
\sum_{i=1}^{n} \beta^i(a) \mathbb{D}^{\alpha}_{a^+} z^i(a^+, \cdot) & = & 0 & \text{in} & (0,T), \\
I^{1-\alpha}_{a^+} z^1(b_1^-, \cdot) & = & 0 & \text{in} & (0,T), \\
I^{1-\alpha}_{a^+} z^i(b_i^-, \cdot) & = & 0 & \text{in} & (0,T), \ i = 2, \ldots, m \\
\beta^i(b_i) \mathbb{D}^{\alpha}_{a^+} z^i(b_i^-, \cdot) & = & 0 & \text{in} & (0,T), \ i = m+1, \ldots, n \\
z^i(\cdot, 0) & = & y^{0,i} & \text{in} & (a, b_i), \quad i = 1, \ldots, n \\
z^i_t(\cdot, 0) & = & \tilde{v}_i & \text{in} & (a, b_i), \quad i = 1, \ldots, n,
\end{array}\right. \tag{14}$$

where $\tilde{f}^i \in L^2(Q_i)$ and $\tilde{v}_i \in L^2(a, b_i)$ are given by

$$\tilde{f}^i = f^i - w^i_{tt} - \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} w^i) - q^i w^i, i = 1, \cdots, n, \tag{15a}$$

$$\tilde{v}_i = v_i - w^i_t(\cdot, 0), \qquad i = 1, \cdots, n. \tag{15b}$$

Theorem

Let $1/2 < \alpha < 1$, $f \in L^2(0,T;\mathbb{L}^2)$, $(g_i)_{i=2\cdots m} \in \left(H^2(0,T)\right)^{m-1}$ *and* $(h_i)_{i=m+1\cdots n} \in \left(H^2(0,T)\right)^{n-m}$ *be such that* $g_i(0) = h_i(0) = 0$, $i = 1,\cdots,n$, $y^0 = (y^{0,i}) \in D(\mathcal{A})$ *and* $v = (v_i) \in \mathbb{L}^2$, *then the problem* (5) *admit a unique weak solution* $y = (y^i) \in \mathcal{C}([0,T],\mathbb{V}) \cap \mathcal{C}^1([0,T],\mathbb{L}^2)$ *given by*

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{\tilde{v}_n}{\sqrt{\mu_n}} \sin(\sqrt{\mu_n} t) + y_n^0 \cos(\sqrt{\mu_n} t) + \frac{1}{\sqrt{\mu_n}} \int_0^t \sin(\sqrt{\mu_n}(t-s)) \tilde{f}_n(s)\,\mathrm{d}s \right\} \zeta_n + w(x,t),$$

Moreover, there exist $\Delta_1, \Delta_2 = C\left(\alpha, \overline{b}, \underline{b}, \overline{\beta}, \overline{q}, \underline{\beta^0}, n\right) > 0$ such that

$$\begin{aligned} \|y\|_{\mathcal{C}([0,T];\mathbb{H}_a^\alpha)} &\leq \Delta_1 \Big(\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \|y^0\|^2_{\mathbb{H}_a^\alpha} + \|\overline{g}\|^2_{\mathcal{C}([0,T])} \\ &+ \|\overline{h}\|^2_{\mathcal{C}([0,T])} + \|\overline{g}_t\|^2_{\mathcal{C}([0,T])} + \|\overline{h}_t\|^2_{\mathcal{C}([0,T])} \Big) \end{aligned} \tag{16a}$$

$$\begin{aligned} \|y'\|_{C([0,T],\mathbb{L}^2)} &\leq \Delta_2 \Big(\|f\|^2_{L^2((0,T);\mathbb{L}^2)} + \|v\|^2_{\mathbb{L}^2} + \|y^0\|^2_{\mathbb{H}_a^\alpha} + \|\overline{g}\|^2_{\mathcal{C}([0,T])} \\ &+ \|\overline{h}\|^2_{\mathcal{C}([0,T])} + \|\overline{g}_t\|^2_{\mathcal{C}([0,T])} + \|\overline{h}_t\|^2_{\mathcal{C}([0,T])} \Big) \end{aligned} \tag{16b}$$

1. Introduction and motivation

2. Fractional Sturm–Liouville wave equations

3. Existence of minimizers and optimality conditions
   characterization of the optimal control.

We are interested in solving the following optimal control problem:

$$\min_{v \in \mathcal{U}_{ad}} \mathcal{J}(v), \tag{17}$$

where

$$\mathcal{J}(v) := \frac{1}{2} \left\| y(v;T) - z_d^0 \right\|_{\mathbb{L}^2}^2 + \frac{1}{2} \left\| y_t(v;T) - z_d^1 \right\|_{\mathbb{L}^2}^2 + \frac{N}{2} \left\| v \right\|_{\mathbb{L}^2}^2 . \tag{18}$$

We have the following existence result of optimal controls.

Theorem

Let $1/2 < \alpha < 1$. Let $q^i \in \mathcal{C}(a, b_i)$ and $\beta^i \in \mathcal{C}^1([a, b_i])$ satisfy Assumption 1. Then, there exists a unique solution $\hat{u} \in \mathcal{U}_{ad}$ of the optimal control problem (17)-(18).

steps of the proof

- We write

$$\mathcal{J}(v) = \frac{1}{2}\pi(v,v) - L(v) + \frac{1}{2}\left\|y(0;T) - z_d^0\right\|_{\mathbb{L}^2}^2 + \frac{1}{2}\left\|y_t(0;T) - z_d^1\right\|_{\mathbb{L}^2}^2 \tag{19}$$

- We show that $\pi(\cdot,\cdot)$ symmetric bilinear continious and coercive functional and that $L(\cdot)L$ is a continuous linear functional.
- We conclude using Lax-Milgram Theorem that there exists a unique $u \in \mathcal{U}_{ad}$ that minimizes $\mathcal{J}$.

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3. Existence of minimizers and optimality conditions
   - characterization of the optimal control.

Theorem

Let $1/2 < \alpha < 1$. Let $q^i \in \mathcal{C}([a, b_i])$ and $\beta^i \in \mathcal{C}^1([a, b_i])$ satisfy Assumption 1. Let $u = (u_i)_i \in \mathcal{U}_{ad}$ be the optimal control for the minimization problem (17)-(18). Then, there exists a unique $p \in L^2((0,T);\mathbb{V}) \cap \mathcal{C}^1([0,T];\mathbb{L}^2)$ such that the triple $(\hat{y}, u, p)$ satisfies

$$\begin{cases} \hat{y}^i_{tt} + \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+}\hat{y}^i) + q^i\hat{y}^i = f^i, & \text{in} \quad Q_i,\ i = 1,\dots,n, \\ I^{1-\alpha}_{a^+}\hat{y}^i(a^+,\cdot) = I^{1-\alpha}_{a^+}\hat{y}^j(a^+,\cdot) & \text{in} \quad (0,T),\ i \neq j = 1,\dots,n, \\ \sum_{i=1}^{n} \beta^i(a)\mathbb{D}^{\alpha}_{a^+}\hat{y}^i(a^+,\cdot) = 0, & \text{in} \quad (0,T), \\ I^{1-\alpha}_{a^+}\hat{y}^1(b_1^-,\cdot) = 0, & \text{in} \quad (0,T), \\ I^{1-\alpha}_{a^+}\hat{y}^i(b_i^-,\cdot) = g^i, & \text{in} \quad (0,T),\ i = 2,\dots,m \\ \beta^i(b_i)\mathbb{D}^{\alpha}_{a^+}\hat{y}^i(b_i^-,\cdot) = h_i, & \text{in} \quad (0,T),\ i = m+1,\dots,n, \\ \hat{y}^i(\cdot,0) = y^{0,i}, & \text{in} \quad (a,b_i),\ \ i = 1,\dots,n, \\ \hat{y}^i_t(\cdot,0) = u^i, & \text{in} \quad (a,b_i),\ \ i = 1,\dots,n, \end{cases} \tag{20}$$

and

$$\begin{cases} p^i_{tt} + \mathcal{D}^{\alpha}_{b_i^-}(\beta^i \mathbb{D}^{\alpha}_{a^+} p^i) + q^i p^i & = 0, & \text{in} & Q_i,\ i = 1, \dots, n, \\ I^{1-\alpha}_{a^+} p^i(a^+, \cdot) & = I^{1-\alpha}_{a^+} p^j(a^+, \cdot) & \text{in} & (0,T),\ i \neq j = 1, \dots, n, \\ \sum_{i=1}^{n} \beta^i(a) \mathbb{D}^{\alpha}_{a^+} p^i(a^+, \cdot) & = 0, & \text{in} & (0,T), \\ I^{1-\alpha}_{a^+} p^i(b_i^-, \cdot) & = 0, & \text{in} & (0,T),\ i = 1, \dots, m \\ \beta^i(b_i) \mathbb{D}^{\alpha}_{a^+} p^i(b_i^-, \cdot) & = 0, & \text{in} & (0,T),\ i = m+1, \dots, n, \\ p^i(\cdot, T) & = \hat{y}^i_t(\cdot, T) - z_d^{1,i} & \text{in} & (a,b),\ i = 1, \dots, n \\ p_t(\cdot, T) & = -(\hat{y}^i(\cdot, T) - z_d^{0,i}) & \text{in} & (a, b_i),\ i = 1, \dots, n \end{cases} \tag{21}$$

and

$$\sum_{i=1}^{n} \int_a^{b_i} \left(u_i + p^i(\cdot, 0)\right)(v_i - u_i)\,\mathrm{d}x \geq 0 \tag{22}$$

for all $v = (v_i)_i \in \mathcal{U}_{ad}$.

steps of the proof

- We write the Euler-Lagrange first order optimality condition for u:

$$\lim_{\theta\to 0}\frac{\mathcal{J}(u+\theta(v-u))-\mathcal{J}(u)}{\theta}\geq 0,\quad \forall v\in\mathcal{U}_{ad}, \tag{23}$$

- it implies

$$\langle z(v-u;T), y(u;T)-z_d^0\rangle_{\mathbb{L}^2}+\langle z_t(v-u;T), y_t(u;T)-z_d^1\rangle_{\mathbb{L}^2}+ N(u, v-u)_{\mathbb{L}^2}\geq 0,\quad \forall v\in\mathcal{U}_{ad}, \tag{24}$$

where $z=z(v-u)$ is the solution of

$$\begin{cases} z_{tt}+\mathcal{A}y = 0 & \text{for} \quad t\in(0,T),\\ z(\cdot,0) = 0, & \\ z_t(\cdot,0) = v-u, & \end{cases} \tag{25}$$

- We then interpret (24), by multiplying (25) by p solution of (21), to get equation (22).

- we show the existence and uniqueness of solution of a sturm-Liouville wave equation on a star graph
- we show the existence and uniqueness of the minimizer of the cost function.
- we assume that $1/2 < \alpha \leq 1$.

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Thanks for your attention