asimodes-error3232Spectral GeometryDoc-Start

Steklov eigenproblems: excursions in numerical analysis, spectral geometry and applications.

Nilima Nigam
(Simon Fraser University)

11ième Biennale Française de Mathématiques Appliquées et Industrielles
Guadalupe 2023

SMAI
société de mathématiques appliquées et industrielles

My talk plan

- Some background
- Discretization
- Spectral Geometry
- Spectral methods
- Steklov-Helmholtz eigenvalues

Vladimir Andreevich Steklov

V. A. Steklov in the 1920s.

Figure: Vladimir A. Steklov, 1864-1926. Picture possibly at the Physical-Mathematical institute in Leningrad.

See [Kutznetsov et al., Notices of the AMS Jan. 2014]

The Steklov Eigenvalue Problem

Find non-trivial s_j and λ_j so that for a given non-negative bounded weight φ

$$-\Delta s_j = 0 \text{ in } \Omega, \qquad \partial_\nu s_j = \lambda_j \varphi s_j \text{ on } \partial\Omega =: \Gamma.$$

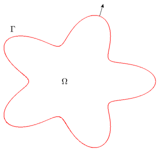


Figure: Bounded domain Ω with boundary Γ

From Steklov's DSc dissertation, and from a talk at the Kharkov Mathematical Society (1895). See also [W. Stekloff], *Sur les problèmes fondamentaux de la physique mathematique, Annales sci. ENS, Sér.* 3, 19 (1902).

Connections to other spectral problems

$\Omega \in \mathbb{R}^2$ a bounded domain, Lipschitz boundary Γ Neumann EVP

$$-\Delta u = \mu u \quad \text{in}\, \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on}\, \Gamma.$$

Vibrations of free membrane

Steklov EVP with $\varphi = 1$

$$-\Delta s = 0 \quad \text{in}\, \Omega, \quad \frac{\partial s}{\partial n} = \lambda s \text{ on}\, \Gamma.$$

Vibrations of a free membrane, mass concentrated on boundary.

Steklov EVP and the Dirichlet-to-Neumann map

Let Ω be a bounded domain in a complete Riemannian manifold, boundary Γ is Lipschitz or smoother.
Consider the Dirichlet problem for given $u \in H^{1/2}(\Gamma)$: find $\mathcal{E}u \equiv U \in H^1(\Omega)$ so that

$$-\Delta U = 0, \quad \text{in}\, \Omega, \quad \frac{\partial U}{\partial n} = u\, \text{on}\, \Omega.$$

This defines the (non-local) Dirichlet-Neumann mapping $\mathcal{D} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. If $\mathcal{S}$ is the single layer operator

$$\mathcal{S} := Tr \circ (-\Delta)^{-1} \circ Tr^*, \quad \mathcal{S} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

then

$$\mathcal{D}(u) = \mathcal{S}^{-1}(u) = \left(\frac{\partial}{\partial n} (\mathcal{E} u) \right)_{\Gamma}.$$

If Γ is smooth, $\mathcal{D}$ is an elliptic $\psi - do$ of order 1, with principle symbol same as $\sqrt{-\Delta_\Gamma}$.

Immediate consequence

Isospectrality:

$$\mathcal{D}u = \lambda u \Leftrightarrow -\Delta s = 0 \quad \text{in}\,\Omega, \quad \frac{\partial s}{\partial n} = \lambda s \text{ on}\,\Gamma.$$

The Steklov EVP is a spectral problem for $\mathcal{D}$, so:

- Steklov spectrum is *discrete* if the trace map $H^1(\Omega) \to H^{1/2}(\Gamma)$ is continuous and the embedding $H^1(\Omega) \to H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact.
- Steklov eigenvalues for $-\Delta$ are non-negative: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots.$
- Similar identification of spectral problems for *DtN* maps for other operators (Lamé, Helmholtz, Maxwell) and analogous Steklov problems

The Sloshing eigenvalue problems

'Sloshing' eigenvalue problem for inviscid,incompressible and heavy liquid

Figure 1. High spot in a coffee cup.

Figure 2. High spot in a snifter.

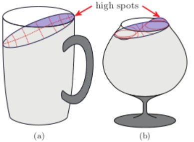


Figure 3. A schematic sketch of location of high spots in a coffee cup (a) and in a snifter (b).

Figure: Figures from Kuznetsov et. al., Notices of the AMS, 61(1), 2014

- The Sloshing/Steklov-Neumann EVP: Find s_j, λ_j so that

$$-\Delta s_j = 0 \text{ in } \Omega, \qquad \partial_\nu s_j = \lambda_j s_j \text{ on } \Gamma_S, \quad \partial_\nu s_j = 0 \text{ on } \Gamma_N.$$

Some background

A (highly incomplete!!) list of references

- Finite element methods: Andre'ev and Todorov ('04), Bi, Li and Yang ('16), Bramble and Osborn '72
- Spectral methods: An, Bi and Luo ('16), Alhejaili and Kao ('19)
- (Non self-adjoint case) Liu, Sun and Turner '18
- (Biharmonic) Antunes and Gazzola ('13)
- Levitin,Parnovski, Polterovich and Sher, *Sloshing, Steklov and corners: Asymptotics of Steklov eigenvalues for curvilinear polygons*, '22.
- Levitin,Parnovski, Polterovich and Sher, *Sloshing, Steklov and corners: Asymptotics of sloshing eigenvalues*, '20
- Free boundary minimal surfaces: Kao, Osting, Oudet ('21)

(Very Incomplete) Background

There is a considerable literature in numerical analysis devoted to the Steklov problem. Partial list:

- Conforming finite elements: Bramble '72.
- Non-conforming finite element methods: Qin Li
- Isoparametric elements Andreev 2004
- Multilevel methods Xie, '14
- Adaptive finite element methods Garau and Morin, '11
- Virtual elements: Mora,Rivera, Rodriguez, '15 .
- Optimal order estimates for Steklov: Bermúdez '00.
- *A posteriori* error estimates for linear elements are derived in Armentano '08, Fei et. al. '21
- Boundary element methods for Steklov: Han '92, Wang et al. '17
- Conformal mapping: Ahlejaili and Kao, '19

(Very incomplete) Background contd.

- Boundary integral operators on curved polygons: Costabel '83, Costabel and Stephan, '83
- Nyström collocation approaches: Kress '89, Colton & Kress '83
- Boundary element approaches: Hsiao & Wendland '77
- Guaranteed eigenvalue enclosures: You, Xie & Liu, '19
- Optimization: Akhmetgaliyev, Kao, Osting '15, Dominguez, N., Shariari, '15, Ammari, Imeri& N., '20
- Spectrum of layer potentials: Helsing & Perfekt, '18
- Recursively Compressed Inverse Preconditioning for 2nd-kind BIE: Helsing, '13
- Steklov eigenfunctions for Robin BVP: Auchmuty, '04, '18
- Stekloff problems for Maxwell: Camano, Lackner and Monk '17
- Steklov for Lamé: Levitin, Monk and Selgas '20, Dominguez '21

Dirichlet v/s Steklov

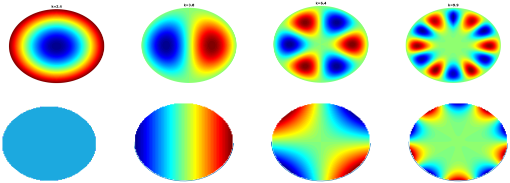


Figure: 1st, 2nd, 3rd and 12th eigenfunctions. Top: Dirichlet. Bottom: Steklov

Eigenvalues of disk v/s kite for the Steklov EVP

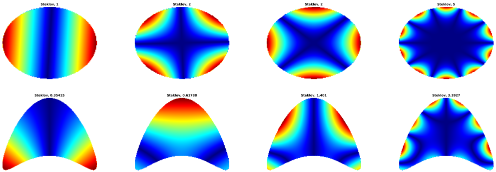


Figure: Steklov eigenfunctions on disk (top) and 'kite' (bottom).

Exact Steklov eigenfunctions on the unit disk are

$$u_j(r,\theta) = r^j \exp(\iota j\theta), \qquad j = 0, 1, \ldots.$$

Rapid decay of u_j away from (smooth) boundary as j increases. Always? [Bruno and Galkowski, '19]

Asymptotics for Steklov eigenvalues

Rozenblyum '86 and Guillemin–Melrose '93: On simply connected, smooth domains,

$$\lambda_{2j} = \lambda_{2j+1} + \mathcal{O}(j^{-\infty}) = \frac{2\pi}{|\Gamma|} j + \mathcal{O}(j^{-\infty}).$$

Girouard-Parnovski-Polterovich-Sher '13: On multiply-connected smooth domains with $\partial\Omega = \cup_{i=1}^{k} \Gamma_i$

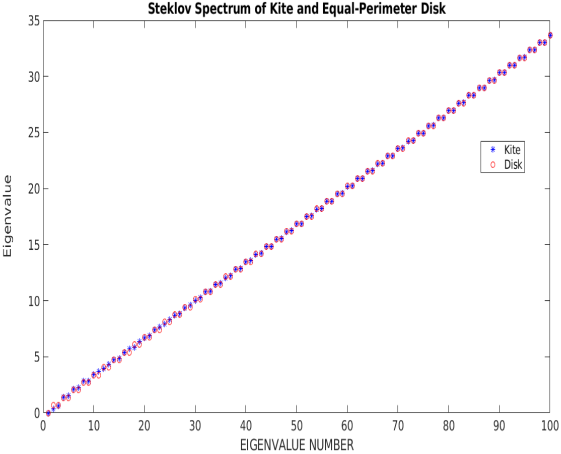
$$\lambda_j = R_j + \mathcal{O}(j^{-\infty})$$

where $R_j = j$th term in $\cup$ Steklov spectra of disks, radii$=|\Gamma_i|/(2\pi)$.
Agranovich, '06: Piecewise C^1 boundaries:

$$\lambda_m = \frac{\pi m}{|\partial\Omega|} + o(m), \qquad m \to \infty$$

Disk v/s kite for the Steklov EVP

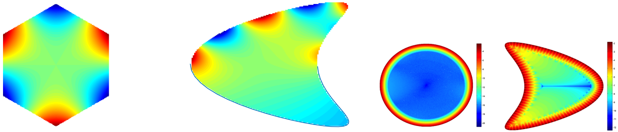
The Steklov spectra of the equal-perimeter disk and kite rapidly become indistinguishable



How to compute Steklov/Sloshing eigenpairs?

For the general sloshing problem (Γ_N is empty for Steklov problems)

$$-\Delta s_j = 0, \qquad \partial_\nu s_j|_{\Gamma_S} = \lambda s_j|_{\Gamma_S}, \quad \partial_\nu s_j|_{\Gamma_N} = 0.$$



Behaviour of Steklov eigenpairs demonstrates the need for high accuracy computations.

Boundary integral reformulation

$$-\Delta s_j = 0, \qquad \partial_\nu s_j|_{\Gamma_S} = \lambda s_j|_{\Gamma_S}, \quad \partial_\nu s_j|_{\Gamma_N} = 0.$$

Define the integral operators $\mathcal{S} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $\mathcal{T} : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ as

$$\begin{aligned} \mathcal{S}[\phi](x) &:= Tr \circ (-\Delta)^{-1} \circ Tr^*[\phi] = \int_\Gamma G(x,y)\phi(y)ds(y) \\ \mathcal{T}[\phi](x) &:= \int_\Gamma \frac{\partial G(x,y)}{\partial n(x)}\phi(y)ds(y), \qquad x \in \Gamma, \end{aligned}$$

Indirect boundary integral formulation of EVP via Single Layer ansatz with density φ:

$$s_j(x) = \int_\Gamma G(x,y)\varphi_j(y)\,dy = \tilde{\mathcal{S}}[\phi_j](x), \quad x \in \Omega.$$

Jump relations

Recall jump relations for single layer if Γ is *Lipschitz*: if the (interior) angle at x_0 is $\omega(x_0)$, then

$$\lim_{\Omega \ni x \to x_0 \in \Gamma} \left(\int_\Gamma G(x,y)\varphi(y)\,dy \right) = \int_\Gamma G(x_0,y)\varphi(y)\,dy = \mathcal{S}[\phi](x_0)$$

$$\lim_{\Omega \ni x \to x_0 \in \Gamma} \nabla \left(\int_\Gamma G(x,y)\varphi(y)\,dy \right) \cdot n_{x_0} = (\mathcal{T}[\phi](x_0) + \omega(x_0)\mathcal{I}[\phi](x_0))$$

So: find (ϕ, λ) so that

$$(\mathcal{T} + \omega(x_0)\mathcal{I})[\phi](x_0) = \lambda \mathcal{S}[\phi](x_0)$$

- Volumetric eigenvalue problem for (u, λ) converted to eigenproblem on Γ for (ϕ, λ).
- Reconstruct u via $\tilde{\mathcal{S}}(\phi)$ where needed

Caution in 2-D

- $\mathcal{S}$ is a Fredholm of index 0 on $H^{-1/2}(\Gamma)$, (Costabel '88)
- $\mathcal{S}$ only injective in $\mathbb{R}^2$ if Ω has analytic capacity $\neq 1$.
- Use modified single layer potential (Kress, '99). Define $\bar{\varphi} := \frac{1}{|\Gamma|} \int_\Gamma \varphi(y) ds_y$.
  Steklov problem: Find λ, φ so that

$$(\mathcal{T} + \omega(x)\mathcal{I})[\varphi - \overline{\varphi}] = \lambda \left(\mathcal{S}[\varphi - \overline{\varphi}] + \overline{\varphi}\right), x \in \Gamma$$

Eigendensity vs/ boundary trace

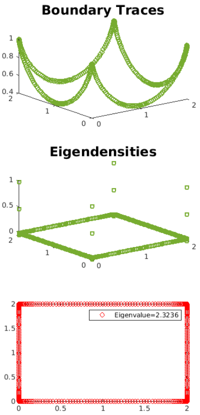


Figure: Sample computed eigendensity ϕ, post-processed boundary trace u

Pure Steklov problems on smooth domains, $\omega \equiv \frac{1}{2}$

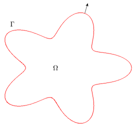


Figure: Bounded planar domain Ω with smooth boundary Γ, whose parametrization is available.

Integral operators will have logarithmic singularities. Use a Nyström collocation approach + specialized quadrature rule (Kussmaul, '69, Martensen '63, Colton and Kress '89)

$$\int_0^{2\pi} \log\left(4\sin^2\frac{t-\tau}{2}\right) f(\tau)\,d\tau \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t) f(t_j), \quad 0 \le t \le 2\pi,$$

Thm: Error for Steklov eigenvalues decreases spectrally with N

Steklov eigenvalues on a disk

Consider the Steklov problem on the unit disk.
True eigenvalues are known to be $0, 1, 1, 2, 2, ...k, k....$
Eigendensities form a trigonometric basis, BIE solver returns exact results for the circle.

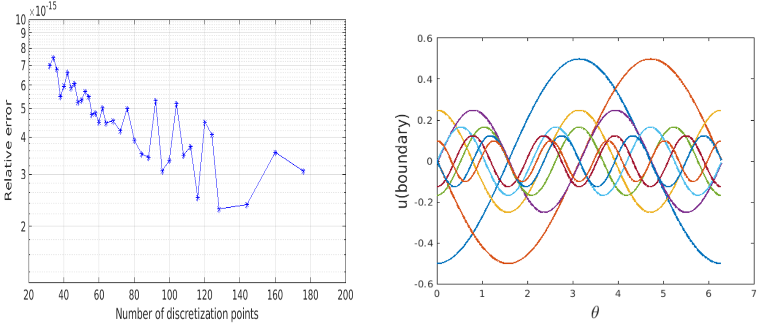


Figure: L: Sum of max errors of first 16 eigenvalues on disk as a function of number of grid points. R: Boundary traces of eigenfunctions

The kite

First 32 Steklov eigenvalues of the kite-shaped domain
Kite: $(x, y) = \{(cos(t) + 0.65cos(2t) - 0.65, 1.5sin(t)), t \in [0, 2\pi\}$

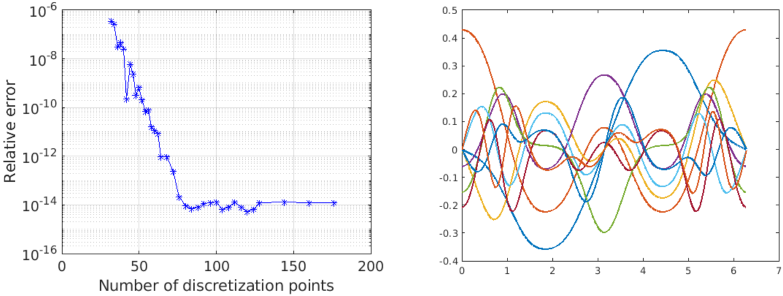
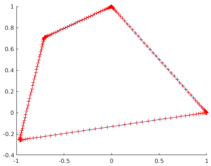


Figure: L: Maximum errors for the first 16 Steklov eigenvalues of a kite-shaped domain. R: Boundary traces of eigenfunctions

Corners/Sloshing

The regularity of the eigenfunctions will depend on geometric features/ the presence of data junctions.



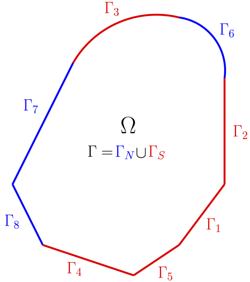
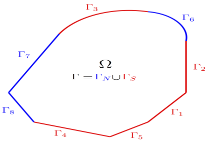


Figure: L: Pure Steklov on polygons. R: Sloshing problems, with Steklov on red curves, Neumann on blue

Sloshing eigenmodes



Let Γ_S and Γ_N meet at an angle of $\alpha\pi$. Komarenko '80:

Regularity of Steklov-Neumann eigenfunctions, angle $\alpha\pi$

Weight spaces: Eigenfunctions $u \in W^2_{2,2+\epsilon}$ restricted to a small neighbourhood of the angular point.
Sobolev spaces: In a neighbourhood of an angular point, eigenfunctions $u \in C^k$ and $u \in W_2^{k+1}$, where

- $k = [\frac{1}{\alpha}]$, if $\frac{1}{\alpha} \notin \mathbb{Z}$
- $k = \frac{1}{\alpha} - 1$ if $\frac{1}{\alpha} \in \mathbb{Z}$.

More recent: behaviour of sloshing modes near the junction, Levitin et.al, 2017, 2019.

Polynomially graded meshes

Collocation on polynomially graded meshes: (Kress '90). The true Steklov eigenvalues of the square are available on a square: (Girouard and Polterovitch, '14)

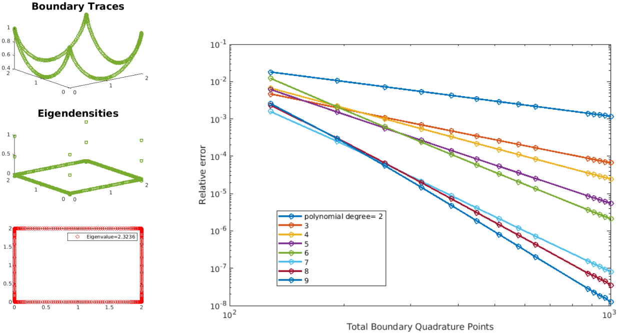
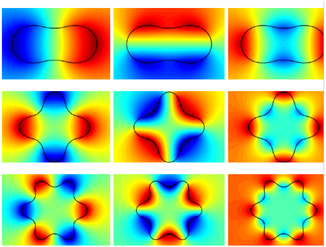


Figure: R: Sum of relative errors in first 6 eigenvalues on square. Choice of polynomial degree in graded mesh

Spectral shape optimization for Steklov

Our approach was used by Akhmetgaliyev, Kao, Osting '15 to numerically show amongst domains with fixed-perimeter smooth boundaries:

- Domains maximizing *pth* Steklov eigenvalue have $p - fold$ symmetry
- Maximizers are unique
- Multiplicity of eigenvalue =2 for even p, multiplicity $= 3$ for odd $p \geq 3$



Geometry of the nodal sets

Our spectral solver allows for exploration of zero level sets of Steklov eigenfunctions. Bruno and Galkowski '20 demonstrate (motivated by the numerics) there are smooth domains for which high-λ eigenfunctions may not have nodal sets dense at $\mathcal{O}(\lambda^{-1})$.

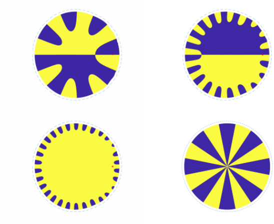


Figure: Top: sample nodal domains on perturbation of a disk. Bottom right: nodal domains at similar frequency for the disk. Note the dramatic change in the geometry of the nodal sets.

Kites and disks, G-P-P-S '13

Let $(x, y) = \{(cos(t) + \kappa cos(2t) - \kappa, 1.5sin(t)), t \in [0, 2\pi]\}$. How fast in κ do the lower eigenvalues

$$\lambda_j \to R_j$$

where $R_j = j$th term in Steklov spectrum of disk of same perimeter?

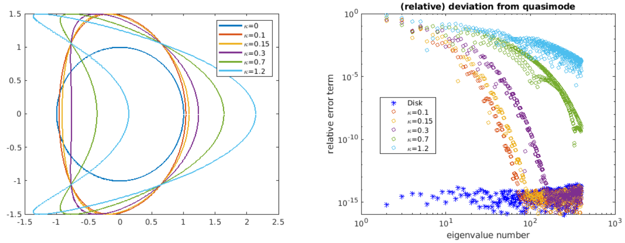


Figure: Impact of changing κ on $|\lambda_j - R_j|/|R_j|$

Steklov eigenfunctions for shape analysis

"... we provide a practical and mathematically justified spectral approach to extrinsic geometry for geometry processing, via an extrinsic alternative to the intrinsic Laplace–Beltrami operator... a surface-only approach to volume-aware shape analysis"...

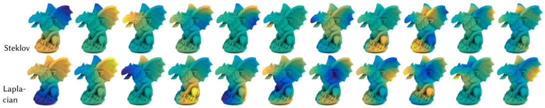
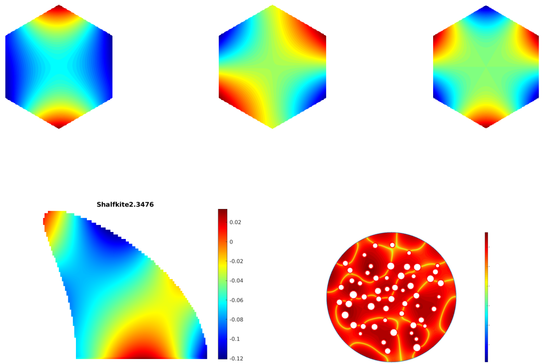


Fig. 13. The Steklov eigenfunctions corresponding to the smallest 11 nonzero eigenvalues, compared to the surface Laplacian eigenfunctions; this model contains 50K triangles and 25K vertices.

Figure: Steklov eigenfunctions on gargoyles v/s surface Laplacian eigenfunctions. Wang, Ben-Chen, Polterovich, & Solomon, '17

Spectra and domain geometry



Quasimodes on Polygons

Levitin-Parnovski-Polterovich-Sher '21: On curvilinear polygons with (angles θ, edges ℓ), $\exists$ (computable) quasimodes σ_j

$$\lambda_j = \sigma_j(\theta, \ell) + \mathcal{O}(j^{-\epsilon}), \quad \forall \epsilon \in (0, \epsilon_0).$$

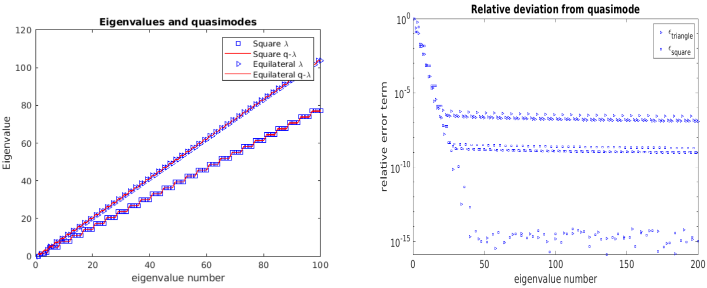


Figure: Comparison of square and equilateral triangle with quasimodes. L: actual σ_j, λ_j. R: Relative deviation

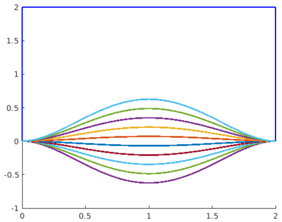
Curvilinear polygons

Levitin-Parnovski-Polterovich-Sher '21: For curvilinear polygons,

$$\lambda_j = \sigma_j(\theta, \ell) + \mathcal{O}(j^{-\epsilon}), \quad \forall \epsilon \in (0, \epsilon_0),$$

quasimodes σ_j depending on angles/edge lengths. Impact of curvature at corners?

Consider side-2 square, bottom edge $= \kappa x^2(2-x)^2/8$.



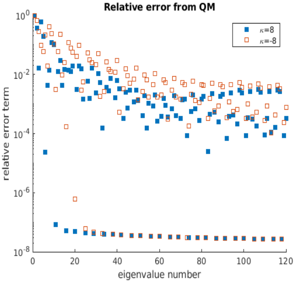
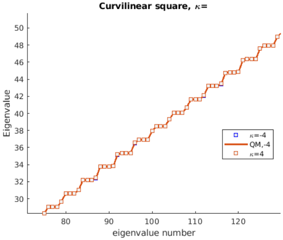


Figure: L: Curvilinear domains. R: Deviation of σ from λ_j, $\kappa = \pm 8$.



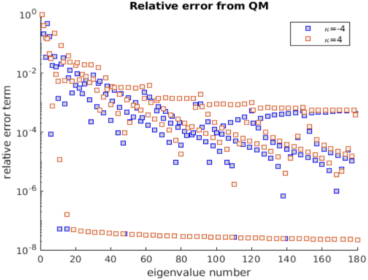
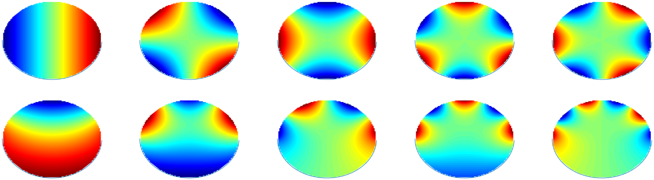


Figure: L: Eigenvalues and quasimodes. R: Deviation of σ from λ_j, $\kappa = \pm 4$.

Comparison: Steklov and sloshing eigenfunctions

Row 1: First 5 non-constant eigenfunctions, pure Steklov.

Row 2: First 5 non-constant eigenfunctions, Steklov-Neumann.



Optimizing a Sloshing-Green's Function

Let $S_{\Gamma_S}(x^*, y)$:=sloshing Green's function in Ω at frequency λ^* and source point $x^* \in \Omega$

$$-\Delta_x S_{\Gamma_S}(x^*, y) = \delta(|x^* - y|), \quad y \in \Omega,$$
$$\frac{\partial}{\partial \nu_x} S_{\Gamma_S}(x^*, y) = \lambda^* S_{\Gamma_S}(x^*, y), \ on\ \Gamma_S,$$
$$\frac{\partial}{\partial \nu_x} S_{\Gamma_S}(x^*, y) = 0, \quad on\ \Gamma_N.$$

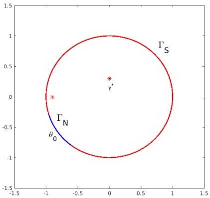


Figure: What is θ_0 and the length $|\Gamma_N|$?

Goal: For fixed λ^*, x^*, y^*, find *center* and *length* of Γ_N to optimize $|S_{\Gamma_S}(x^*, y^*)|$.

Can use the sloshing eigenfunctions

(Ammari, Imeri, N. '20) Let $\lambda^\star$ be close to sloshing eigenvalue λ_j (with multiplicity m). Let $\{s_k\}_{k=1}^m$ be the $L^2-(\Gamma_s)$ normalized sloshing eigenfunctions.

$$-\Delta_x u_k = 0,\ \frac{\partial}{\partial \nu_x} s_k = \lambda_j s_k,\ \text{on}\, \Gamma_S, \frac{\partial}{\partial \nu_x} s_k = 0, \text{on}\, \Gamma_N,$$

Then the sloshing Green's function can be written as

$$S_{\Gamma_S}(x^*, y^*) = -\frac{1}{2\pi}\log|x^* - y^*| + \sum_{k=1}^{m} \frac{s_k(x^*)s_k(y^*)}{\lambda^* - \lambda_j} + R(\lambda^*, x^*, y^*)$$

The remainder R is analytic in λ^*.

Optimization for unit disk

Unit circle with $\lambda_\star = 2.5$, $x^\star = (-0.9, 0)^{\mathrm{T}}$, $y \in \{(0, r)^{\mathrm{T}} \in \mathbb{R}^2 \mid r > 0\}$. $\mathrm{S}^{\lambda_\star}_{\mathrm{Steklov}}(x^\star, y) \equiv$ Steklov function and $\mathrm{S}^{\lambda^\star}_{\mathrm{End}}(x^\star, y) \equiv$ Steklov-Neumann function on optimized domain. $\theta_0 \in [0, 2\pi)$ and l_{N} represent the (angular) location and length of Neumann boundary.

	$r = 0.1$	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 0.9$
$\mathrm{S}^{\lambda_\star}_{\mathsf{Steklov}}(x^\star, y)$	-0.022	-0.048	-0.147	-0.332	-0.492
$\mathrm{S}^{\lambda_\star}_{\mathsf{End}}(x^\star, y)$	12.67	39.83	90.50	-133.6	-200.4
θ_0	0.42π	0.42π	0.39π	1.96π	1.96π
l_{N}	0.36π	0.36π	0.36π	0.36π	0.36π
$\left\lvert \frac{\mathrm{S}^{\lambda_\star}_{\mathsf{End}}(x^\star, y)}{\mathrm{S}^{\lambda_\star}_{\mathsf{Steklov}}(x^\star, y)} \right\rvert$	586	838	615	402	407

Optimization

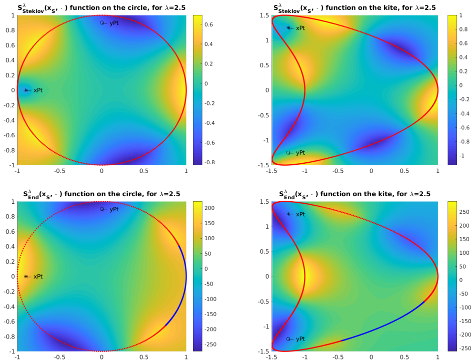


Figure: Top row: Green's function for pure Steklov problem, $\lambda^* = 2.5$. Bottom row: optimized Green's function. Neumann boundary along blue edge.

Steklov eigenfunctions are remarkable! I

Under (mild) assumptions on domain boundary Γ, can define an H^1-equivalent inner-product $\langle \cdot\, , \cdot \rangle_\partial$ through

$$\langle v, w \rangle_\partial := \int_\Omega \nabla v \cdot \nabla w \; + \; \int_{\partial\Omega} v\, w\,,$$

for $v, w \in H^1(\Omega)$, where we use the trace theorem.

- The Steklov eigenfunctions s_j can be normalized as $\|s_j\|_\partial = 1$
- The s_j are mutually $\langle \cdot\, , \cdot \rangle_\partial$ orthogonal:

$$\int_\Omega \nabla s_i \cdot \nabla s_j = \lambda_j \int_{\partial\Omega} s_i\, s_j = \lambda_i \int_{\partial\Omega} s_i\, s_j = 0\,, \qquad \text{for } i \neq j\,,$$

due to Green's identity.

Steklov eigenfunctions are remarkable! II

- We also have that traces of s_j are L^2- orthogonal on the boundary. In fact,

$$\begin{aligned} \langle s_i\,, s_i \rangle_\partial &= (1+\lambda_i)\|s_i\|^2_{L^2(\partial\Omega)} = 1\,, && \text{for } i \in \mathbb{N}_0\,, \\ \int_\Omega \nabla s_i \cdot \nabla s_i &= \lambda_i \int_{\partial\Omega} (s_i)^2 = \frac{\lambda_i}{1+\lambda_i}\,, && \text{for } i \in \mathbb{N}_0\,, \\ s_0 &\equiv \frac{1}{\sqrt{|\partial\Omega|}}\,. \end{aligned}$$

- $\{s_j\}$ form an orthogonal basis for harmonic functions in $H^1(\Omega)$

A Steklov-spectral method for Robin problems I

[Auchmuty, 2011, Auchmuty and Cho JCAM 2017]
Consider the Robin boundary value problem

$$\Delta u(x) = 0 \quad \text{in } \Omega\,, \quad \partial_{\nu_y} u(y) + b\, u(y) = g(y) \text{ on } \Gamma\,,$$

where $g \in H^{-1/2}(\Gamma)$, $b > 0$ a constant.
Taking an L^2-inner product of Robin condition on Γ,

$$\int_\Gamma [\partial_{\nu_y} u(y) + b\, u(y)] s_j(y)\, d\Gamma = [\lambda_j + b] \int_\Gamma Tr[u] s_j\, d\Gamma = \int_\Gamma g s_i.$$

Suppose $u(x) = \sum_{i=0}^{\infty} u_j s_j(x)$.

A Steklov-spectral method for Robin problems II

Then we have for the exact solution u to the Robin problem with boundary data $g^{(\mathrm{R})}$ that

$$u(x) = \sum_{j=0}^{\infty} \frac{(1+\lambda_i)}{b+\lambda_j}\, s_j(x) \int_\Gamma g\, s_i\, d\Gamma$$

Let $s_i(x) = \mathcal{S}[\varphi_i]$ where φ_i is the eigendensity, and $\varphi_{i,N} := \sum_{n=-N}^{N} c_n^{(i)} \exp(\mathrm{i}\, nt)$. Let $\tilde{s}_i = \mathcal{S}[\tilde{\phi}_i]$ and then

$$\tilde{u}_{M,N}^{(\mathrm{R})} = \sum_{i=0}^{M} \frac{(1+\lambda_i)}{\lambda_i + b}\, \tilde{s}_i \int_{\partial\Omega} g^{(\mathrm{R})}\, \tilde{s}_i\, .$$

A Steklov-spectral method for Robin problems III

Discretization error [Imeri-N (2021)]

Theorem (Imeri-N. '22)

Let $\partial\Omega \in C^p$ with $p \in \mathbb{N}$, $p \geq 2$ and let $g^{(\mathrm{D})} \in H^p(\Gamma)$. Then if the exact solution to the Robin problem with boundary data $g^{(\mathrm{R})}$ is u, we have

$$\|u - \tilde{u}^{(\mathrm{R})}_{M,N}\|_\delta = \mathcal{O}\Big(\frac{1}{M^p}\Big) + \mathcal{O}\Big(\frac{\log(M)}{N^{p-1/2}}\Big).$$

Robin data: $\partial_{\nu_y} u(y) + 1.5\, u(y) = log(|y - (0, 2.5))$

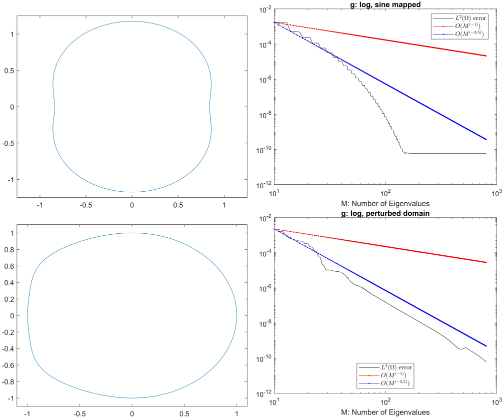


Figure: Domains of interest. Top Left: Sine perturbation of. Bottom Left: Perturbed disk (C^2 but not C^3). Right column: L^2 errors in M, number of Steklov modes. Blue: Theoretical rate. Black: Computed. Red: M^{-1}

The Steklov-Helmholtz problem

(Joint with K. Patil)
Let $\Omega \subset \mathbb{R}^2$ be bounded. Let μ be a given wavenumber.
The Steklov-Helmholtz problem is to find u_j, σ_j so that

$$-\Delta u_j - \mu^2 u_j = 0 \text{ in } \Omega, \qquad \frac{\partial}{\partial \nu} u_j = \sigma_j u_j \text{ on } \partial\Omega =: \Gamma.$$

Some interesting complexities

- μ^2 could be an interior Dirichlet eigenvalue
- μ^2 could be an interior Neumann eigenvalue
- μ could be complex, in which case the problem isn't self-adjoint.

In this talk, $\mu \in \mathbb{R}$ is not a Laplace eigenvalue.

Steklov-Helmholtz eigenvalues on a disk

On a disk of radius R, the Steklov-Helmholtz eigenfunctions are

$$u_n(r,\theta) = J_n(\mu r) exp(in\theta).$$

The Steklov-Helmholtz eigenvalues $\sigma_0, \sigma_1, \sigma_1, \sigma_2, \sigma_2, ...$are therefore given analytically as

$$\sigma_n = \mu \frac{J_n'(\mu R)}{J_n(\mu R)}, n = 0, 1, 2..$$

Convergence study on a disk

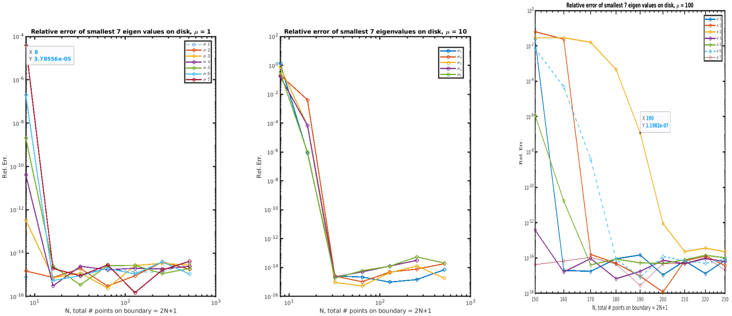


Figure: Relative errror in first 7 eigenvalues v/s number of points on boundary. L: $\mu = 1$. M: $\mu = 10$. R: $\mu = 100$

Can you 'see' the shape?

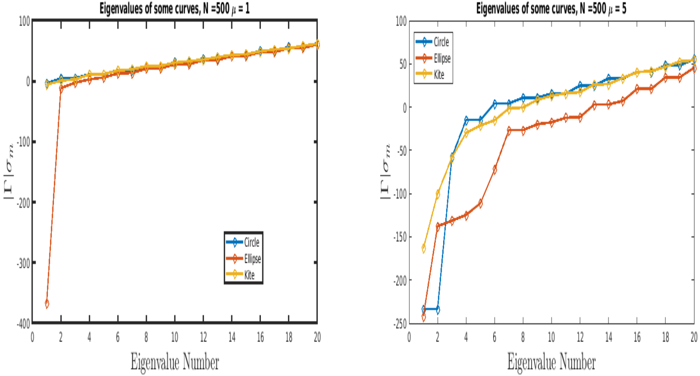
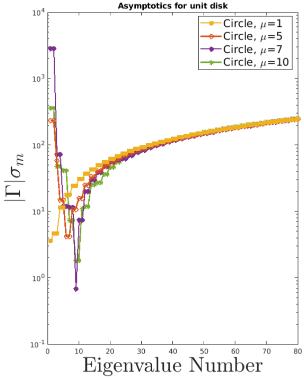


Figure: Circle, ellipse, kite. L: $\mu = 1$. R: $\mu = 5$.

Asymptotics of $|\Gamma||\sigma_m|$



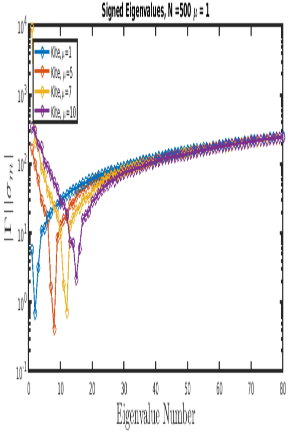


Figure: Asymptotic behavior, changing wavenumber μ. L: Disk. R: Kite.

Density, boundary trace and eigenfunctions

Unit disk, $\mu = 1$ and $\mu = 7$

Disk, $\mu = 1$

Disk, $\mu = 7$

Density, boundary trace and eigenfunctions

Ellipse, $\mu = 1$ and $\mu = 7$

Ellipse, $\mu = 1$

Ellipse, $\mu = 7$

The final slide

Steklov eigenproblems are fascinating!
Many thanks for listening.

Steklov eigenfunctions are remarkable! I

Recall: the Steklov eigenvalues $\lambda_0, \lambda_1, \ldots$ and respective Steklov eigenfunction $s_0, s_1, \ldots$ solve

$$\begin{cases} \triangle s_j(x) = 0 & \text{in } \Omega\,, \\ \partial_{\nu_y} s_j(y) = \lambda_j\, s_j(y) & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

Under (mild) assumptions on domain boundary, can define an H^1-equivalent inner-product $\langle \cdot\,, \cdot \rangle_\partial$ through

$$\langle v, w \rangle_\partial := \int_\Omega \nabla v \cdot \nabla w \; + \int_{\partial\Omega} v\, w\,,$$

for $v, w \in H^1(\Omega)$, where we use the trace theorem.

- The Steklov eigenfunctions s_j can be normalized as $\|s_j\|_\partial = 1$

Steklov eigenfunctions are remarkable! II

- The s_j are mutually $\langle \cdot \, , \cdot \rangle_\partial$ orthogonal:

$$\int_\Omega \nabla s_i \cdot \nabla s_j = \lambda_j \int_{\partial\Omega} s_i \, s_j = \lambda_i \int_{\partial\Omega} s_i \, s_j = 0 \, , \qquad \text{for } i \neq j \, ,$$

  due to Green's identity.

- We also have that traces of s_j are L^2- orthogonal on the boundary. In fact,

$$\langle s_i \, , s_i \rangle_\partial = (1 + \lambda_i) \| s_i \|^2_{L^2(\partial\Omega)} = 1 \, , \qquad \text{for } i \in \mathbb{N}_0 \, ,$$

$$\int_\Omega \nabla s_i \cdot \nabla s_i = \lambda_i \int_{\partial\Omega} (s_i)^2 = \frac{\lambda_i}{1 + \lambda_i} \, , \qquad \text{for } i \in \mathbb{N}_0 \, ,$$

$$s_0 \equiv \frac{1}{\sqrt{|\partial\Omega|}} \, .$$

Steklov eigenfunctions are remarkable! III

- $\{s_j\}$ form an orthogonal basis for harmonic functions in $H^1(\Omega)$
- The Steklov-Fourier series $\sum_{j=1}^{\infty} c_j s_j(x)$ converges to a harmonic function in H^1 iff

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty$$

- For arbitrary $u \in H^1(\Omega)$, define the trace

$$Tr[u] := \sum_{j=0} [u, s_j]_\partial \, Tr[s_j]$$

Steklov eigenfunctions are remarkable! IV

- For $s \geq 0$, define

$$H^s(\Gamma) := \{f \in L^2(\Gamma) | \sum_{j=0} [1+\lambda_j]^{2s} |f_j|^2 < \infty\},$$

where

$$f_j := \sqrt{1+\lambda_j} \int_\Gamma f(x)\, Tr[s_j](x)\, d\Gamma$$

- For $s < 0$, define $H^s(\Gamma)$ as the completion of $L^2(\Gamma)$ wrt same norm as above

Steklov eigenfunctions are remarkable! V

- Can define $E_H : L^2(\Gamma) \to L^2(\Omega)$, the 'harmonic extension' operator

$$E_H[g](x) := \sum_{i=0}^{\infty} (1 + \lambda_i)\, s_i(x) \int_\Gamma g\, s_i\,.$$

For convergence in $H^1(\Omega)$, we'd need

$$\sum_{i=0}^{\infty} [(1 + \lambda_i) \int_\Gamma g\, s_i\,]^2 < +\infty$$

so need $g \in H^{1/2}(\Gamma)$.

Some recent literature ...

- Auchmuty, *Steklov eigenproblems and the representation of solutions of elliptic BVP,* Numerical Functional Analysis and Optimization, 2011
- Auchmuty and Cho, *Steklov approximations of harmonic BVP on planar regions,* JCAM 2017.
- Girouard and Polterovich, *Spectral geometry of the Steklov problem*, J. Spectral Theory 7 (2017)
- Akhmetgaliyev, Kao, Osting, *Computational Methods For Extremal Steklov Problems,* SIAM J. Control and Optimization, 55 (2016)
- Levitin, Parnovski, Polterovich, Sher, *Sloshing, Steklov and corners: Asymptotics of Steklov eigenvalues for curvilinear polygons*, arXiV, 2019
- Ammari, Imeri, Nigam, *Optimization of Steklov-Neumann eigenvalues,* J.Computational Physics, 406, (2020)
- Bruno and Galkowski, *Domains without dense Steklov nodal*

Convergence for a 'sloshing' problem

Find u_j, λ_j so that

$$-\Delta u_j = 0 \text{ in } \Omega, \qquad \frac{\partial}{\partial \nu} u_j = \lambda_j u_j \text{ on } \Gamma_S, \quad \frac{\partial}{\partial \nu} u_j = 0 \text{ on } \Gamma_N.$$

Single layer approach + Polynomial grading

8 ALEXANDRE GIROUARD AND IOSIF POLTEROVICH

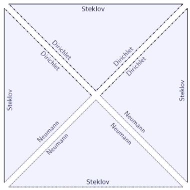


FIGURE 2. Decomposition of the Steklov problem on a square into four mixed problems on a triangle.

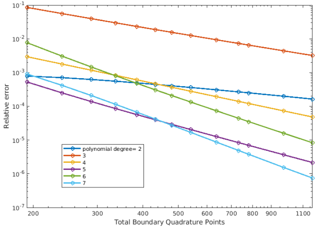


Figure: The Steklov spectrum on a square can be decomposed into 4 (mixed) problems (Girouard and Polterovich, '17). Sum of relative errors for first 10 eigenvalues.

Convergence and quasimodes

For a unit square, the quasimodes are

$$\sigma_{4m} = \sigma_{4m-1} = \sigma_{4m-2} = \sigma_{4m-3} = (m - 1/2)\pi, \qquad m \in \mathbb{N}.$$

Quasimodes are useful for checking accuracy of methods, in the absence of 'true' eigenvalues.

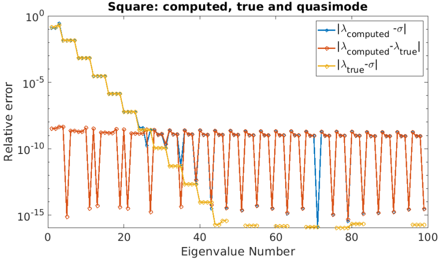


Figure: Comparison of computed eigenvalues on square with $\lambda_{true,j}$ and quasimodes σ_j

Computational strategy

Parametrize Γ as $x(t) = (x_1(t), x_2(t))$ $(0 \leq t \leq 2\pi)$.
Define $\psi(\tau) := \varphi(x(\tau))$ for $(0 \leq \tau \leq 2\pi)$.
Let

$$r(t,\tau) = |x(t) - x(\tau)|,$$

the BIE system is written as: find (λ, ψ) so that

$$\int_0^{2\pi} L(t,\tau)(\psi(\tau) - \overline{\psi})d\tau + \frac{1}{2}(\psi(t) - \overline{\psi}) =$$

$$= \lambda \left(\int_0^{2\pi} K(t,\tau)(\psi(\tau) - \overline{\psi})d\tau + \overline{\psi} \right)$$

where

$$L(t,\tau) = \frac{1}{2\pi} \frac{(x_2'(t)[x_1(t) - x_1(\tau)] - x_1'(t)[x_2(t) - x_2(\tau)])}{r^2(t,\tau)} |x'(\tau)^2|,$$

$$K(t,\tau) = \frac{1}{2\pi} \log\left[r(t,\tau)\right] |x'(\tau)^2| \ .$$

Polynomially graded meshes

Collocation on polynomially graded meshes: (Kress '90). On each of the intervals $a_q \leq \tau \leq b_q$, $q = 1, \ldots, Q_S$, suppose parametrization is $(x(\tau), y(\tau))$.
Use a polynomial change of variables of the form $\tau = w_q(s)$, where

$$w_q(s) = a_q + (b_q - a_q)\frac{[v(s)]^p}{[v(s)]^p + [v(2\pi - s)]^p}, \quad 0 \leq s \leq 2\pi,$$

$$v(s) = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\pi - s}{\pi}\right)^3 + \frac{1}{p}\frac{s - \pi}{\pi} + \frac{1}{2},$$

and where $p \geq 2$ is an integer. Each function w_q is smooth and increasing in the interval $[0, 2\pi]$, and their k-th derivatives satisfy $w_q^{(k)}(0) = w_q^{(k)}(2\pi) = 0$ for $1 \leq k \leq p - 1$.