Error estimates of a theta-scheme for second-order mean field games

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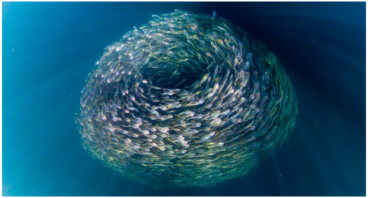
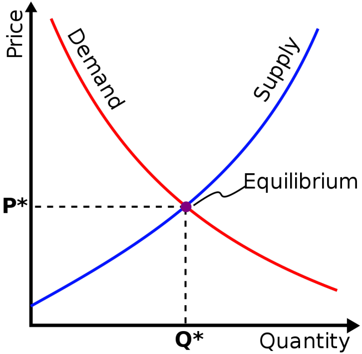
Mean field games (MFGs)

2006: Lasry-Lions, Huang-Malhame-Caines.

a. Traffic congestion



b. Fish migration



c. Supply-demand-pricing model

An example of N players symmetric differential games [1]

1. The dynamic of each player: For $i = 1, \dots, N$,

$$dX_t^i = v_t^i dt + \sigma dW_t^i, \quad X_0^i \sim m_0.$$

Here, v_t^i is the strategy (drift), and W_t^i is the independent Brownian motion (volatility).

[1] Example from [F. Delarue cours de PGMO 22].

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Here, v_t^i is the strategy (drift), and W_t^i is the independent Brownian motion (volatility).

2. The payoff:

$$J^i = \mathbb{E}\Big[\int_0^T \underbrace{\frac{1}{2}|v_t^i|^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{N-1}\sum_{j \neq i} f(X_t^i - X_t^j)}_{\text{potential energy}} dt + \underbrace{g(X_T^i)}_{\text{terminal cost}} \Big]$$

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Pass to the limit

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The **interaction** term: Let $m_{-i}^N(t) = \frac{1}{N-1}\sum_{j\neq i} \mathrm{Dirac}_{X_t^j}$, then

$$\frac{1}{N-1}\sum_{j\neq i} f(X_t^i - X_t^j) = f * m_{-i}^N(t, X_t^i).$$

Its mean field limit is

$$f * m(t,x).$$

Mean field games

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2. Given a distribution $m(t,x)$, minimize over v:

$$J_m(v) = \mathbb{E}\Big[\int_0^T \underbrace{\frac{1}{2}|v_t|^2}_{\text{kinetic energy}} + \underbrace{f * m(t, X_t^v)}_{\text{potential energy}} dt + \underbrace{g(X_T)}_{\text{terminal cost}}\Big]$$

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3. Nash equilibrium: $(\bar{v}, \bar{m})$, such that

$$\bar{v} = \operatorname{argmin} J_{\bar{m}}(v); \quad (1)$$

$$\bar{m}(t, \cdot) = \operatorname{law}(X_t^{\bar{v}}). \quad (2)$$

Hamilton-Jacobi-Bellman equation

The first problem (1) is a stochastic optimal control problem:

$$\begin{cases} \inf_v \mathbb{E}\Big[\int_0^T \underbrace{\frac{1}{2}|v_t|^2}_{\text{kinetic energy}} + \underbrace{f * \bar{m}(t, X_t^v)}_{\text{potential energy}} dt + \underbrace{g(X_T)}_{\text{terminal cost}}\Big] \\ \text{s.t. } dX_t^v = v_t dt + \sigma dW_t, \quad X_0^v \sim m_0. \end{cases} \tag{3}$$

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Define the value function $\bar{u}\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}$,

$$\begin{cases} \bar{u}(t, x) = \inf_v \mathbb{E}\Big[\int_t^T \underbrace{\frac{1}{2}|v_\tau|^2}_{\text{kinetic energy}} + \underbrace{f * \bar{m}(\tau, X_\tau^v)}_{\text{potential energy}} d\tau + \underbrace{g(X_T)}_{\text{terminal cost}} \Big] \\ \text{s.t. } dX_\tau^v = v_\tau d\tau + \sigma dW_\tau, \forall \tau \in [t, T], \quad \text{and } X_t^v = x. \end{cases} \tag{4}$$

Explanation: the optimal value from time t and state x.

Hamilton-Jacobi-Bellman equation

The **HJB equation** associated to problem (3) is:

$$\begin{cases} -\frac{\partial \bar{u}}{\partial t}(t,x) - \frac{\sigma^2}{2}\Delta_x \bar{u}(t,x) + \frac{1}{2}\left|\frac{\partial \bar{u}}{\partial x}(t,x)\right|^2 = f * \bar{m}(t,x); \\ \bar{u}(T,x) = g(x). \end{cases} \tag{5}$$

Hamilton-Jacobi-Bellman mapping:

$$\mathbf{HJB}(\bar{m}) := \bar{u},$$

where $\bar{u}$ satisfies (5).

Optimal strategy

The optimal strategy $\bar{v}$ is given by

$$\bar{v}(t,x) = -\frac{\partial \bar{u}}{\partial x}(t,x). \tag{6}$$

Optimal control mapping:

$$\mathbf{V}(\bar{u}) := -\frac{\partial \bar{u}}{\partial x}.$$

Fokker-Planck equation

The second problem (2) is the distribution of the solution of the following SDE:

$$dX_t^{\bar{v}} = \bar{v}_t dt + \sigma dW_t, \quad X_0^{\bar{v}} \sim m_0.$$

The distribution of $X_t^{\bar{v}}$ satisfies the following **Fokker-Planck** equation:

$$\begin{cases} \frac{\partial \bar{m}}{\partial t}(t,x) - \frac{\sigma^2}{2}\Delta_x \bar{m}(t,x) + \mathrm{div}(\bar{v}\bar{m}(t,x)) = 0; \\ \bar{m}(0,x) = m_0(x). \end{cases} \tag{7}$$

Fokker-Planck mapping:

$$\mathbf{FP}(\bar{v}) := \bar{m},$$

where $\bar{m}$ satisfies (7).

MFGs equations

Introduce the value function $\bar{u}$ by (4).

The Nash equilibrium of MFGs:

$$\begin{cases} \bar{u} = & \mathbf{HJB}(\bar{m}); \\ \bar{v} = & \mathbf{V}(\bar{u}); \\ \bar{m} = & \mathbf{FP}(\bar{v}). \end{cases}$$

Equivalent to the following forward-backward PDEs by (5)-(7):

$$\begin{cases} -\frac{\partial \bar{u}}{\partial t}(t,x) - \frac{\sigma^2}{2}\Delta_x \bar{u}(t,x) + \frac{1}{2}\left|\frac{\partial \bar{u}}{\partial x}(t,x)\right|^2 = f * \bar{m}(t,x); \\ \bar{v}(t,x) = -\frac{\partial \bar{u}}{\partial x}(t,x); \\ \frac{\partial \bar{m}}{\partial t}(t,x) - \frac{\sigma^2}{2}\Delta_x \bar{m}(t,x) + \mathrm{div}(\bar{v}\,\bar{m}(t,x)) = 0; \\ \bar{u}(T,x) = g(x),\ \ \bar{m}(0,x) = m_0(x). \end{cases} \tag{8}$$

General second-order MFGs

Notations.

- $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$;
- $Q := [0,1] \times \mathbb{T}^d$;
- $\mathcal{D} := \left\{\mu \in \mathbb{L}^2(\mathbb{T}^d) \,|\, \mu \geq 0, \int_{\mathbb{T}^d} \mu(x)dx = 1\right\}$.

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Data.

- Running cost $\ell^c \colon Q \times \mathbb{R}^d \to \mathbb{R}$;
- Congestion cost $f^c \colon Q \times \mathcal{D} \to \mathbb{R}$;
- Initial condition $m_0^c \in \mathcal{D}$;
- Terminal cost $g^c \colon \mathbb{T}^d \to \mathbb{R}$.

General second-order MFGs

The equation of second-order MFGs on a torus: $\forall\,(t,x) \in Q$,

$$\begin{cases} -\frac{\partial u}{\partial t}(t,x) - \sigma\Delta u(t,x) + H^c\left(t,x,\nabla u(t,x)\right) = f^c(t,x,m(t)); \\ v(t,x) = -H^c_p\left(t,x,\nabla u(t,x)\right); \\ \frac{\partial m}{\partial t}(t,x) - \sigma\Delta m(t,x) + \operatorname{div}(vm(t,x)) = 0; \\ u(1,x) = g^c(x),\;\; m(0,x) = m^c_0(x), \end{cases} \tag{MFG}$$

where the Hamiltonian H^c is the Fenchel conjugate of ℓ^c:

$$H^c(t,x,p) = \sup_{v\in\mathbb{R}^d} -\langle p, v\rangle - \ell^c(t,x,v).$$

Assumptions

Assumption A.

1. (Lipschitz regularity). There exists $L^c > 0$, such that
   - $\ell^c(\cdot,\cdot,v)$, $\ell^c_v(\cdot,x,v)$, $g^c(\cdot)$ and $f^c(\cdot,\cdot,m)$ are L^c-Lipschitz continuous;
   - $f^c(t,x,\cdot)$ is L^c-Lipschitz continuous w.r.t. $\|\cdot\|_{\mathbb{L}^2}$-norm.
2. (Strong convexity). Function $\ell^c(t,x,\cdot)$ is α^c-strongly convex.
3. (Monotonicity). For any m_1 and m_2 in $\mathcal{D}$,

$$\int_{\mathbb{T}^d}\Big(f^c(t,x,m_1)-f^c(t,x,m_2)\Big)\big(m_1(x)-m_2(x)\big)dx \geq 0.$$

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$$\int_{\mathbb{T}^d} \Big(f^c(t,x,m_1) - f^c(t,x,m_2) \Big) \big(m_1(x) - m_2(x) \big) dx \geq 0.$$

Assumption B.

Equation (MFG) has a unique solution (u^*, v^*, m^*), with

$$u^*, m^* \in \mathcal{C}^{1+r/2, 2+r}(Q) \text{ and } v^* \in \mathcal{C}^r(Q) \cap \mathbb{L}^\infty([0,1]; \mathcal{C}^{1+r}(\mathbb{T}^d)),$$

for some $r \in (0,1)$.

Classical solution of MFGs

Theorem 1 [Bonnans-L.-Pfeiffer]

Let Assumption A hold. Suppose that

- there exists $C > 0$, such that

$$\ell^c(t, x, v) \leq C\|v\|^2 + C, \qquad |f^c(t, x, m)| \leq C;$$

- $\ell^c \in \mathcal{C}^3(Q \times \mathbb{R}^d)$ and $m_0^c, g^c \in \mathcal{C}^3(\mathbb{T}^d)$.

Then, (MFG) has a unique solution (u^*, v^*, m^*) satisfying Assumption B for any $r < 1$.

Some references on numerical methods of MFGs

- **Finite difference** method: Implicit schemes [Achdou-Capuzzo 10, Achdou-Camilli-Capuzzo 13, Achdou-Porretta 16].

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- **Deep learning** method: DeepFBSDE [Germain-Pham-Warin 21, Carmona-Lauriere 21, Germain-Lauriere-Pham-Warin 22].

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Notations in finite difference scheme

- Natural canonical basis of $\mathbb{R}^d$: $(e_i)_{i=1,\ldots,d}$;
- Time step: $\Delta t = 1/T$; Time set $\mathcal{T} = \{0, 1, \ldots, T-1\}$.
- Space step: $h = 1/N$; Discretization of $\mathbb{T}^d$:

$$S = \big\{(i_1, i_2, \ldots, i_d)h \;\mid\; i_1, \ldots, i_d \in \mathbb{Z}/N\mathbb{Z}\big\}.$$

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Operators for the centered finite difference scheme:

Let $\mu\colon S \to \mathbb{R}$ and $\omega\colon S \to \mathbb{R}^d$.

- Discrete gradient: $\nabla_h \mu = \left(\frac{\mu(\cdot+he_i)-\mu(\cdot-he_i)}{2h}\right)_{i=1}^{d}$;
- Discrete Laplacian: $\Delta_h \mu = \sum_{i=1}^{d} \frac{\mu(\cdot+he_i)+\mu(\cdot-he_i)-2\mu(\cdot)}{h^2}$;
- Discrete divergence: $\mathrm{div}_h \omega = \sum_{i=1}^{d} \frac{\omega_i(\cdot+he_i)-\omega_i(\cdot-he_i)}{2h}$.

Notations for the discretization of the data of MFGs

Notations.

- $\mathbb{R}(\mathbb{T}^d)$ (resp. $\mathbb{R}(S)$): Set of functions from $\mathbb{T}^d$ (resp. S) to $\mathbb{R}$.
- Lattice:

$$B_h(x) := \prod_{i=1}^{d} [x - he_i/2, x + he_i/2), \quad \forall x \in S.$$

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Two operators.

- $\mathcal{I}_h \colon \mathbb{R}(\mathbb{T}^d) \to \mathbb{R}(S)$,

$$\mathcal{I}_h(m^c)(x) = \int_{B_h(x)} m^c(y)dy, \quad \forall x \in S;$$

- $\mathcal{R}_h \colon \mathbb{R}(S) \to \mathbb{R}(\mathbb{T}^d)$,

$$\mathcal{R}_h(m)(y) = \frac{m(x)}{h^d}, \quad \forall x \in S,\ y \in B_h(x).$$

Discretization of the data of MFGs

- Running cost
$$\ell(t,x,v) := \ell^c(t\Delta t, x, v);$$
- Hamiltonian
$$H(t,x,p) := H^c(t\Delta t, x, p);$$
- Initial condition
$$m_0(x) := \mathcal{I}_h(m_0^c)(x);$$
- Terminal cost
$$g(x) := g^c(x);$$
- Congestion cost
$$f(t,x,m) := \frac{1}{h^d} \int_{y \in B_h(x)} f^c\Big(t\Delta t, y, \mathcal{R}_h(m)\Big)\, dy.$$

Remark: Compared to [Achdou-Camilli-Capuzzo 2013], we do not introduce a "numerical" Hamiltonian for the discretization of Hamiltonian.

The θ-scheme of heat equation

Consider the heat equation in Q:

$$\begin{cases} \frac{\partial m}{\partial t}(t,x) - \Delta m(t,x) = 0, & (t,x) \in Q; \\ m(0,x) = m_0(x), & x \in \mathbb{T}^d. \end{cases} \quad (9)$$

Let $\theta \in [0,1]$, the θ-scheme of (9): For all $t \in \mathcal{T}$,

$$\frac{m(t+1) - m(t)}{\Delta t} - \theta \Delta_h m(t+1) - (1-\theta)\Delta_h m(t) = 0.$$

Remark:

1. $\theta = 0$: The explicit scheme,

$$\frac{m(t+1) - m(t)}{\Delta t} - \Delta_h m(t) = 0.$$

2. $\theta = 1$: The implicit scheme,

$$\frac{m(t+1) - m(t)}{\Delta t} - \Delta_h m(t+1) = 0.$$

The θ-scheme of heat equation

The θ-scheme of (9): For all $t \in \mathcal{T}$,

$$\frac{m(t+1) - m(t)}{\Delta t} - \theta \Delta_h m(t+1) - (1-\theta)\Delta_h m(t) = 0.$$

It is equivalent to

$$\begin{cases} \frac{m(t+1/2) - m(t)}{\Delta t} - (1-\theta)\Delta_h m(t) & = 0, \\ \frac{m(t+1) - m(t+1/2)}{\Delta t} - \theta \Delta_h m(t+1) & = 0, \end{cases}$$

where $m(t+1/2)$ is an auxiliary variable determined by $m(t)$.

The θ-scheme of (MFG): θ-MFG

1. HJB equation:

$$\begin{cases} -\frac{u(t+1)-u(t+1/2)}{\Delta t} - \theta\sigma\Delta_h u(t+1/2) = 0, \\ -\frac{u(t+1/2)-u(t)}{\Delta t} - (1-\theta)\sigma\Delta_h u(t+1/2) + H(\nabla_h u(t+1/2)) = f(m(t)); \end{cases}$$

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4. Initial and terminal conditions:

$$m(0,x) = m_0(x), \qquad u(T,x) = g(x).$$

The CFL condition

Recall in Assumption A:

1. L^c: the Lipschitz constant of the data;
2. α^c: the strong-convexity constant of ℓ^c w.r.t. v.

Define a constant

$$M = \frac{1}{\alpha^c}\Big(2 \max_{(t,x)\in Q} \|\ell^c_v(t,x,0)\| + 3\sqrt{d}L^c\Big).$$

The CFL condition of θ-MFG:

$$\Delta t \leq \frac{h^2}{2d(1-\theta)\sigma}, \qquad h \leq \frac{2(1-\theta)\sigma}{M}. \qquad \text{(CFL)}$$

Main result

Let A_1 and A_2 be two finite sets and $\mu \colon A_1 \times A_2 \to \mathbb{R}$. Define

$$\|\mu\|_{\infty,\infty} := \max_{x \in A_1} \max_{y \in A_2} |\mu(x,y)|, \qquad \|\mu\|_{\infty,1} := \max_{x \in A_1} \sum_{y \in A_2} |\mu(x,y)|.$$

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Theorem 2

Let Assumptions A and B hold true. Let $\theta \in (1/2, 1)$ and let $(\Delta t, h)$ satisfy the condition (CFL). Then, θ-MFG has a unique solution (u_h, v_h, m_h). Moreover, there exists a constant $C > 0$, independent of Δt and h, such that

$$\|u_h - u\|_{\infty,\infty} + \|m_h - m\|_{\infty,1} \leq Ch^{r},$$

where $u(t) := u^*(t\Delta t)$ and where $m(t) := \mathcal{I}_h(m^*(t\Delta t))$.

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Methodology of the numerical analysis

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Remark: We skip the part of (DMFG) in this presentation.

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1. Existence and uniqueness.
   - Existence: Applying Brouwer's fixed point theorem to (DMFG);
   - Uniqueness: A fundamental inequality.
2. Stability analysis.

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   - Regularity of (u^*, v^*, m^*) in Assumption B.

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A perturbed θ-scheme: θ-MFG(δ)

A perturbed version of θ-MFG with additional terms (δ_1, δ_2):

1. Perturbed HJB equation:

$$\begin{cases} -\frac{u(t+1)-u(t+1/2)}{\Delta t} - \theta\sigma\Delta_h u(t+1/2) = \delta_1(t), \\ -\frac{u(t+1/2)-u(t)}{\Delta t} - (1-\theta)\sigma\Delta_h u(t+1/2) + H(\nabla_h u(t+1/2)) = f(m(t)); \end{cases}$$

2. Optimal control:

$$v(t,x) = -H_p(t,x,\nabla_h u(t+1/2,x)).$$

3. Perturbed Fokker-Planck equation:

$$\begin{cases} \frac{m(t+1/2)-m(t)}{\Delta t} - (1-\theta)\sigma\Delta_h m(t) + \mathrm{div}_h(mv(t)) = 0, \\ \frac{m(t+1)-m(t+1/2)}{\Delta t} - \theta\sigma\Delta_h m(t+1) = \delta_2(t). \end{cases}$$

4. Initial and terminal conditions:

$$m(0,x) = m_0(x), \qquad u(T,x) = g(x).$$

Stability analysis of θ-MFG: HJB equation

Suppose that $(u^\delta, v^\delta, m^\delta)$ satisfies θ-MFG(δ) and $m^\delta \geq 0$. Let (u_h, v_h, m_h) be a solution of θ-MFG. Denote by

$$\delta u = u^\delta - u_h, \quad \delta v = v^\delta - v_h, \quad \delta m = m^\delta - m_h.$$

Lemma 1 (Stability of HJB)

Let Assumptions A and condition (CFL) hold true. Then,

$$\|\delta u\|_{\infty,\infty} \leq \frac{L^c}{h^{d/2}} \|\delta m\|_{\infty,2} + \Delta t \|\delta_1\|_{1,\infty}.$$

Recall:

$$\|\mu\|_{\infty,2} = \max_{x \in A_1} \|\mu(x,\cdot)\|_{\ell^2}, \qquad \|\mu\|_{1,\infty} = \sum_{x \in A_1} \|\mu(x,\cdot)\|_{\ell^\infty}.$$

Stability analysis of θ-MFG: Fundamental inequality

Lemma 2 (Fundamental inequality)

Let Assumption A and condition (CFL) hold true. Then,

$$\frac{\alpha}{2}\sum_{t\in\mathcal{T}}\sum_{x\in S}\|\delta v(t,x)\|^2(m_h+m^\delta)(t,x) \leq \sum_{t\in\mathcal{T}}\sum_{x\in S}\delta u(t+1,x)\delta_2(t,x) + \sum_{t\in\mathcal{T}}\sum_{x\in S}\delta m(t,x)\delta_1(t,x).$$

Corollary: Scheme θ-MFG has a unique solution.

Remark: A similar fundamental equality is proved for an implicit scheme in [Achdou-Camilli-Capuzzo 2013]. A continuous version of this fundamental equality is given in [Cardaliaguet-Lasry-Lions-Porretta 2013].

Stability analysis of θ-MFG: Energy inequality

- Summing the two steps of the Fokker-Planck equation in θ-MFG,

$$\frac{m(t+1) - m(t)}{\Delta t} - \theta \Delta_h m(t+1) - (1-\theta)\Delta_h m(t) + \mathrm{div}_h mv(t) = 0.$$

- Let μ satisfy a perturbed Fokker-Planck equation:

$$\begin{cases} \frac{\mu(t+1)-\mu(t)}{\Delta t} - \theta \Delta_h \mu(t+1) - (1-\theta)\Delta_h \mu(t) + \mathrm{div}_h \mu v(t) = \mathrm{div}_h \eta_1(t) + \eta_2(t), \\ \mu(0) = 0. \end{cases}$$

Lemma 3 (Energy inequality)

Let $\theta > 1/2$ and $\|v\|_{\infty,\infty} \leq M$. Then, there exists some constant c independent of h and Δt such that

$$\|\mu\|_{\infty,2}^2 \leq c\Delta t \sum_{t \in \mathcal{T}} \|\eta_1(t)\|_2^2 + \|\eta_2(t)\|_2^2.$$

Consistency error

Recall that (u^*, v^*, m^*) is the solution of (MFG).

- Let $u(t) = u^*(t\Delta t)$ and $m(t) = \mathcal{I}_h(m^*(t\Delta t))$.
- Compute the auxiliary variable $u(t+1/2)$ by

$$-\frac{u(t+1) - u(t+1/2)}{\Delta t} - \theta\sigma\Delta_h u(t+1/2) = 0.$$

- Let $v(t,x) = -H_p(t,x,\nabla_h u(t+1/2))$.

Lemma 4 (Consistency error)

Let Assumption B and condition (CFL) hold. The triplet (u, v, m) is a solution to the perturbed system θ-MFG(δ), with perturbation terms satisfying

$$\delta_1 = \mathcal{O}(h^r),\ \delta_2 = \mathrm{div}_h \eta_1 + \eta_2,\ \eta_1 = \mathcal{O}(h^{2r+d}),\ \eta_2 = \mathcal{O}(h^{r+d}).$$

Sketch of the proof of Theorem 2

Recall the consistency error:

$$\delta_1 = \mathcal{O}(h^r),\ \delta_2 = \mathrm{div}_h \eta_1 + \eta_2,\ \eta_1 = \mathcal{O}(h^{2r+d}),\ \eta_2 = \mathcal{O}(h^{r+d}).$$

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$$\epsilon \le C_2 \left(\|\delta m\|_{\infty,2} h^{r-d/2} + h^{2r} \right).$$

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3. By the energy inequality,
$$\|\delta m\|_{\infty,2}^2 \le C_3 h^d \left(\epsilon + h^{2r} \right).$$
4. Combining the previous two estimates, it follows:
$$\|\delta m\|_{\infty,2} \le C_4 h^{r+d/2}.$$

Table of Contents

Conclusion and perspectives

Conclusion:

- We propose a θ-scheme of second-order MFGs and give its error estimates;
- We propose a general framework of discrete MFGs (not mentioned in this presentation) and give its stability analysis (essentially, the fundamental inequality).

Perspectives:

- Some numerical algorithms to solve θ-MFG (specially in potential case);
- Some "splitting" methods to reduce the complexity of computation in high dimensions, etc.

Some references

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Thank You!