Deterministic Optimal control on Riemannian manifolds under probability knowledge of the initial condition

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Joint work with: H. Zidani and F. Jean

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Outline of the talk

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1. Introduction
2. Setting of the problem
3. Main results
4. Conclusion and perspectives

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  - $X_{.} \in M$ is the state variable and $u(.) \in U$ is the control.
  - Standard hypotheses on the dynamics.

$$\begin{cases} f : M \times U \to TM \text{ is continuous and Lipschitz with respect to the state,} \\ \forall x \in M, \text{ the set of functions } f(., U) := \{f(., u) \,:\, u \in U\} \text{ is convex.} \end{cases}$$

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- We consider the following optimal control problem:

$$\vartheta(t_0, \mu_0) := \begin{cases} \min \int_M \ell(X_T^{t_0,x_0,u}) d\mu_0(x_0), \\ \text{such that (1) holds.} \end{cases}, \ \vartheta \text{ is the value function.}$$

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$$g\sharp\mu \in \mathcal{P}(Y) \ : \ \forall h : Y \to \mathbb{R} \text{ Borel, bounded, } \int_M h \, d(g\sharp\mu) = \int_M h \circ g \, d\mu$$

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- We want to characterize the value function as the unique viscosity solution of an HJB equation of the form

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How to define the Hamiltonian? How to define viscosity notion?

How can we define the derivative $D_\mu v$?

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- The state space in now the space $\mathcal{P}(M)$.

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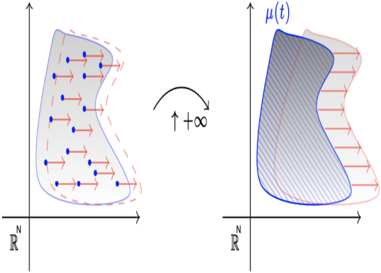
Setting of the problem

- The state space in now the space $\mathcal{P}(M)$.
- Many other interesting applications take place in the space $\mathcal{P}(M)$.

- Multi-agent systems:



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State of the art

Imperfect information on the initial condition

- Deterministic differential games with imperfect information on the initial condition on the space $\mathcal{P}(\mathbb{R}^d)$ done by Quincampoix, Cardaliaguet, ...

Multi-agent systems

- General optimal control problem on the space $\mathcal{P}(\mathbb{R}^d)$ done by Bonnet, Rossi, Frankowska, Marigonda, Quincampoix, Cardaliaguet, Jimenez, Piccoli, ...

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► General theory of viscosity solutions on $\mathcal{P}(M)$: **not done yet.**

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Dynamic programming

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Theorem (Dynamic programming)
Let $\mu \in \mathcal{P}(M)$, $t \in [0,T]$ and $h \in [0,T-t]$. Then it holds

$$\vartheta(t,\mu) = \inf_{u(.)\in U} \{ \vartheta(t+h, \mu_{t+h}^{t,\mu,u}) \}.$$

Wasserstein spaces

- Let (Y, d_Y) be a Polish space (i.e. complete and separable metric space).
- Define

$$\mathcal{P}_2(Y) := \{\mu \in \mathcal{P}(Y) \ : \ \int_Y d_Y^2(x, x_0) d\mu(x) < \infty, \quad \forall\, x_0 \in Y\,\}.$$

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$$d_W^2(\mu, \nu) := \inf_\gamma \Big\{ \int_{Y \times Y} d_Y^2(x, y) d\gamma(x, y) \Big\},$$

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- If Y is compact then $\mathcal{P}(Y) = \mathcal{P}_2(Y)$ and $\mathcal{P}(Y)$ is compact.
- If (Y, d_Y) is Polish, then $(\mathcal{P}_2(Y), d_W)$ is Polish.
- If (Y, d_Y) is a geodesic space, then $(\mathcal{P}_2(Y), d_W)$ is a geodesic space.

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- (M, d_M) and (TM, d_{TM}) are Riemannian manifolds and equipped with their Riemannian distances
- We can define the Wasserstein spaces $\mathcal{P}_2(M)$ and $\mathcal{P}_2(TM)$.
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Rough idea

- $\mathcal{P}_2(TM)$ plays the role of the tangent bundle of $\mathcal{P}(M)$.
- What plays the role of the "tangent space" at a point $\mu \in \mathcal{P}(M)$ is

$$\mathcal{P}_2(TM)_\mu := \Big\{ \gamma \in \mathcal{P}_2(TM) \ : \ \pi^M \sharp \gamma = \mu \Big\},$$

$\pi^M : TM \to M$ is the canonical projection.

[1]References: Lott and Villani (2009), Sturm (2006), Ohta (2009), Gigli (2011),...

Hamiltonian in Wasserstein space

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Key observations

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Key observations

- When the state space is M, the test functions used are C^1.
- We cannot define C^1 functions on $\mathcal{P}(M)$ because there isn't any smooth structure on it.
- However, for Hamiltonians of the type

$$H(\mu, D_\mu v) = \inf_{u \in U} \{ D_\mu v \centerdot (f(.,u)\sharp\mu) \} \quad \mu \in \mathcal{P}(M),$$

  we only need directional derivatives
- We take test functions on $\mathcal{P}(M)$ to be directionally differentiable everywhere.

Viscosity notion in $\mathcal{P}(M)$

Definition (Semiconvex/semiconcave/DC function)

Let $F : \mathcal{P}(M) \to \mathbb{R}$ be a function.

- We say that F is semiconvex if there exists $\lambda \in \mathbb{R}$ such that for every geodesic $\alpha : [0,1] \to \mathcal{P}(M)$ the following inequality holds
$$F(\alpha_t) \leq (1-t)F(\alpha_0) + tF(\alpha_1) - \frac{\lambda}{2}t(1-t)d_W^2(\alpha_0, \alpha_1).$$
- Let $F : \mathcal{P}_2(M) \to \mathbb{R}$ be a function. We say that F is semiconcave if $-F$ is semiconvex.
- We say that F is DC if it can be represented as a difference of semiconvex functions

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$$F(\alpha_t) \leq (1-t)F(\alpha_0) + tF(\alpha_1) - \frac{\lambda}{2}t(1-t)d_W^2(\alpha_0, \alpha_1).$$
- Let $F : \mathcal{P}_2(M) \to \mathbb{R}$ be a function. We say that F is semiconcave if $-F$ is semiconvex.
- We say that F is DC if it can be represented as a difference of semiconvex functions

Theorem (Differentiation of DC functions)
Let $F : \mathcal{P}(M) \to \mathbb{R}$ be a Lipschitz and DC function. Then F admits directional derivatives everywhere. We say that F is differentiable and we denote the differential by $D_\mu F$.

Viscosity notion in $\mathcal{P}(M)$

Theorem (`Squared Wasserstein distance`) Let $\sigma \in \mathcal{P}(M)$. The function $\mu \mapsto d_W^2(\mu, \sigma)$ is semiconcave. Hence it admits directional derivatives everywhere.

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Definition (Test functions)

- Let $\mathcal{TEST}_1$ be the set defined as

$$\mathcal{TEST}_1 := \{(t, \mu) \mapsto \psi(t) + a\, d_W^2(\mu, \sigma) : \ a \in \mathbb{R}^+, \sigma \in \mathcal{P}(M) \text{ and } \psi \in C^1\}$$

- Let $\mathcal{TEST}_2$ be the set defined as

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Remarks

- $\mathcal{TEST}_1$ functions are semiconcave with respect to the measure.
- $\mathcal{TEST}_2$ functions are semiconvex with respect to the measure.

Viscosity notion in $\mathcal{P}(M)$

Definition (Viscosity solutions)

- An u.s.c. function $v : [0,T] \times \mathcal{P}(M) \to \mathbb{R}$ is a viscosity **subsolution** if for all $\phi \in \mathcal{TEST}_1$ such that $v - \phi$ attains a local **max** at (t, μ) we have:

$$-\partial_t \phi + H(\mu, D_\mu \phi) \leq 0.$$

- A l.s.c. function $v : [0,T] \times \mathcal{P}(M) \to \mathbb{R}$ is a viscosity **supersolution** if for all $\phi \in \mathcal{TEST}_2$ such that $v - \phi$ attains a local **min** at (t, μ) we have:

$$-\partial_t \phi + H(\mu, D_\mu \phi) \geq 0.$$

- A continuous function v is said to be a viscosity solution, if it is both a supersolution and a subsolution and verifies the final condition

$$v(T, \mu) = \int_M \ell \, d\mu, \quad \forall \mu \in \mathcal{P}(M).$$

Comparison principle, Well-posedness

Theorem (Comparison principle)
Let $v, w : [0,T] \times \mathcal{P}_2(M) \to \mathbb{R}$ be respectively a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution on $[0,T] \times \mathcal{P}(M)$. Then it holds:

$$\sup_{[0,T]\times\mathcal{P}_2(M)} (v-w)_+ \leq \sup_{\{T\}\times\mathcal{P}_2(M)} (v-w)_+,$$

where $(k)_+ = \max(k, 0)$.

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Theorem (Well-posedness)
The value function ϑ is the unique continuous viscosity solution to

$$\begin{cases} \partial_t v + H(\mu, D_\mu v) = 0, & (t,\mu) \in [0,T) \times \mathcal{P}(M), \\ v(T,\mu) = \int_M \ell \, d\mu. \end{cases}$$

Outline of the talk

Conclusion and perspectives

What is done?

- We defined a new notion of viscosity for Hamilton Jacobi Bellman equations in presence of imperfect information on the initial condition to guarantee well-posedness.
- We defined a framework to study more general Hamilton Jacobi equations in $\mathcal{P}(M)$.

Future work

- Continue the work on more general Hamilton Jacobi equations on $\mathcal{P}(M)$.
- Study more general optimal control problems in $\mathcal{P}(M)$.
- Extend this work to $\mathcal{P}(\mathbb{R}^N)$. **(O.J, A. Prost, H. Zidani)**
- Extend this notion of viscosity to other metric spaces. **(O.J, H. Zidani)**

Thank you for your attention!!!