An isoperimetric problem derived from Gamow's model for the atomic nucleus

Joint work with Michael Goldman and Benoît Merlet

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- $\exists 0 < m_1 \le m_2$ such that[1]
  - $m \le m_1 \implies$ **existence** of minimizers (ball)
  - $m > m_2 \implies$ **non-existence** (fission)

[1] Knüpfer and Muratov 2013, 2014

Gamow's liquid drop model for the atomic nucleus (2)

- Generalizations

$$\min_{E} \quad \left\{ P(E) + \iint_{E\times E} G(x-y)\,\mathrm{d}x\,\mathrm{d}y \ : \ E \subseteq \mathbb{R}^n, |E| = m \right\}$$

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  - ↳ $\forall \alpha \in (0,n)$, $\exists m_0 > 0$, $(m < m_0 \implies [B]_m$ **unique minimizer**$)$
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- General assumptions on G
  1. $G \in L^1(\mathbb{R}^n)$, radial, non-negative
  2. $|x|G(x) \in L^1(\mathbb{R}^n)$ ("fast" decay at ∞), and (up to $G \rightsquigarrow \gamma G$), we set

$$I_G^1 := \int_{\mathbb{R}^n} |x|G(x)\,\mathrm{d}x = \mathbf{K}_n$$

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Reformulation

- Since G is integrable

$$\iint_{E\times E} G(x-y)\,\mathrm{d}x\,\mathrm{d}y = \|G\|_{L^1(\mathbb{R}^n)}|E| - \iint_{E\times(\mathbb{R}^n\setminus E)} G(x-y)\,\mathrm{d}x\,\mathrm{d}y,$$

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$$\Updownarrow \qquad \text{rescaling}$$

$$\min_E \quad \left\{P(E) - \gamma\, P_{G_\varepsilon}(E) \;:\; |E| = |B_1|\right\},$$

where $\varepsilon := \left(\frac{|B_1|}{m}\right)^{1/n}$, and $G_\varepsilon = \varepsilon^{-(n+1)} G(\varepsilon^{-1}\cdot)$.

- $m \to \infty \iff \varepsilon \to 0.$

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$$P_{G_\varepsilon}(E) = \iint\limits_{\substack{x,y \in (B_\varepsilon + \partial E) \\ E \times E^{\mathrm{c}}}} G_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y \quad \simeq \quad \text{perimeter}$$

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$$\Big\downarrow \leq \quad \varepsilon \to 0^{5} \quad \text{(choice of } \mathbf{K}_n\text{)}$$

$$\int_{\mathbb{R}^n} |D \mathbf{1}_E| = P(E)$$

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- $(P - \gamma P_{G_\varepsilon})(E) \xrightarrow{\varepsilon \to 0} (1-\gamma) P(E)$

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Existence and convergence

Large mass minimizers

Theorem (M.P. 2021)

For all $\gamma \in (0,1)$, there exists $\varepsilon_e = \varepsilon_e(n,\gamma,G)$ s.t. $\forall \varepsilon < \varepsilon_e$, $(\star)$ admits a minimizer. In addition, every minimizer E_ε is ***connected*** *and satisfies, up to translation,*

$$B_{1-\delta(\varepsilon)} \subseteq E_\varepsilon \subseteq B_{1+\delta(\varepsilon)}, \quad \text{where } \delta(\varepsilon) \to 0.$$

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- Back to Gamow's problem, this means that if $I^1_G < \mathbf{K}_n$, then there exist minimizers of arbitrarily large mass
- Every minimizer E_ε is a quasi-minimizer of the perimeter, that is,

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  for every F s.t. $E_\varepsilon \triangle F \subset\subset B_r(x)$ with $r < \varepsilon r_0(n,G,\gamma)$.
- $\implies$ (non-uniform) partial regularity[6]: ∂E_ε is loc. $C^{1,\frac{1}{2}}$ away from a set of dimension $\le n-8$

[6] Tamanini 1982, Ambrosio and Paolini 1999, Rigot 2000a,b

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For every $\gamma < 1$, $\exists \varepsilon_* > 0,\ \forall \varepsilon < \varepsilon_*$, $\boldsymbol{B_1}$ is the unique minimizer.

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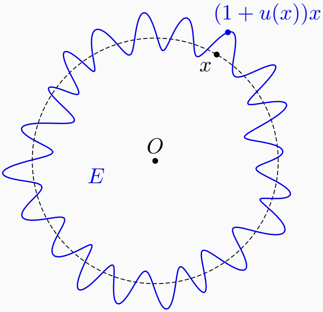
- 1^{st} step : **true** among **nearly spherical sets**



$\|u\|_{\mathbf{Lip}} \leq \alpha$

The planar case

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Proposition (Convexity)

In dimension 2, for every $0 < \gamma < 1$, there exists $\varepsilon_* = \varepsilon_*(G, \gamma)$ s.t. for every $\varepsilon < \varepsilon_*$, any minimizer of $(\star)$ is ***convex****.*

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- For any nearly spherical set E with u s.t. $\|u\|_\infty < \frac{1}{4}$, we have[7]

$$\|\nabla_\tau u\|_\infty \leq 2 \left(\frac{1 + \|u\|_\infty}{1 - \|u\|_\infty} \right) \|u\|_\infty^{\frac{1}{2}},$$

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Theorem (B. Merlet, M.P. – Minimality of the disk)

In dimension 2, for every $0 < \gamma < 1$, the disk is the unique minimizer, up to translations, whenever $\varepsilon < \varepsilon_*$.

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Convexity of minimizers in dimension 2

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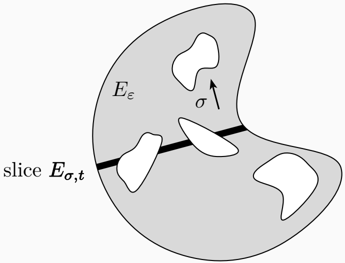

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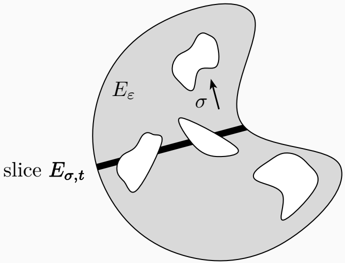

- We rewrite the "**critical functional**" $\mathcal{E}_\varepsilon := P - P_{G_\varepsilon}$ as

$$\mathcal{E}_\varepsilon(E) = \frac{1}{4} \int_{\mathbb{S}^1} \int_{\mathbb{R}} (\mathcal{H}^0 - P^1_\varepsilon)(E_{\sigma,t}) \, \mathrm{d}t \mathcal{H}^1(\mathrm{d}\sigma),$$

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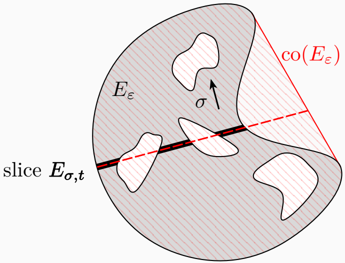

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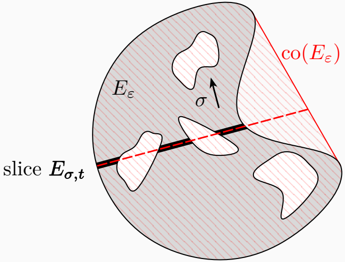

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- We notice $(\mathcal{H}^0 - P^1_\varepsilon)(\text{———}) \leq (\mathcal{H}^0 - P^1_\varepsilon)(\text{– – –})$

$\implies \mathcal{F}_{\gamma,\varepsilon}(\mathrm{co}(E_\varepsilon)) \leq \mathcal{F}_{\gamma,\varepsilon}(E_\varepsilon)$ (recall: $\mathcal{F}_{\gamma,\varepsilon} = \gamma P + (1-\gamma)\mathcal{E}_\varepsilon$)

In higher dimensions ($n \geq 3$)

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- Recall: E_ε is a quasi-minimizer of the perimeter:

$$P(E_\varepsilon; B_r(x)) \leq P(F; B_r(x)) + \frac{\Lambda(n, G, \gamma)}{\varepsilon} |E_\varepsilon \triangle F|,$$

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- **If** we had **uniform** $C^{1,\alpha}$ regularity estimates:

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- **If** we had **uniform** $C^{1,\alpha}$ regularity estimates:
  Hausdorff CV $\implies$ $C^{1,\alpha}$ CV $\implies$ E_ε is the ball

Uniform regularity of minimizers (1)

- De Giorgi's strategy for local minimizers of the perimeter[8]

[8] Revisited by Maggi 2012

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- The (spherical) **excess** of E_ε at $x \in \partial E_\varepsilon$ at scale r is

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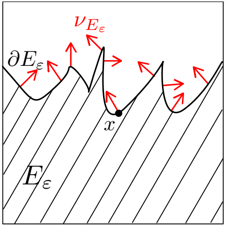

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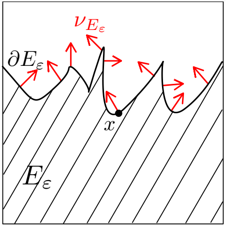

- The excess measures the **variation of the normal vector** to ∂E_ε

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Uniform regularity of minimizers (2)

- Rmk: since $E_\varepsilon \to B_1$ and $\partial E_\varepsilon \to \partial B_1$, we have

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\mathbf{e}(E_\varepsilon, x, r) = 0 \quad \text{uniformly in } x \in \partial E_\varepsilon,$$

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- Then it is enough to show:

Theorem

$\exists\alpha$, ε_0, τ, r_0 such that, if E_ε is a minimizer with $\varepsilon < \varepsilon_0$ satisfying

$$\mathbf{e}(E_\varepsilon, x, r) < \tau, \qquad \text{for some } x \in \partial E \text{ and some } r < r_0,$$

then

$$\mathbf{e}(E_\varepsilon, x, s) \le C(n, G, \gamma)\left(\frac{s}{r}\right)^{2\alpha}, \qquad \forall s \in (0, r).$$

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- By Campanato's criterion, the normal vector is $C^{0,\alpha}$, uniformly as $\varepsilon \to 0$.

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- To bridge the gap between scales ε^η and ε, we use the naive scaling $\mathbf{e}(E_\varepsilon, r) \leq \left(\frac{R}{r}\right)^{n-1} \mathbf{e}(E_\varepsilon, R)$ whenever $r < R$ and the decay of G at ∞

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- Main steps (roughly):
    1. If the excess is small in B_{2r}, $\partial E_\varepsilon \cap B_r$ is almost entirely covered by a 1-Lipshitz graph Γ_u

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$$\int_{\mathbb{R}^{n-1}} \nabla u \cdot \nabla \varphi \qquad + \text{"smaller terms"}$$
$$- \gamma \iint_{D_{2r} \times D_{2r}} (u(x') - u(y'))(\varphi(x') - \varphi(y'))G_\varepsilon(x' - y', 0) = 0$$

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     for every $\varphi \in C_c^1(D_r)$ s.t. $\|\nabla \varphi\|_\infty = 1$.
  3. Since $\varepsilon \ll r$, u is close to a harmonic function

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4. Thus the **flatness** of E_ε at a smaller scale λr is **much smaller** than the excess at scale r: $\mathbf{f}(\lambda r) \lesssim \lambda^2 \mathbf{e}(r)$

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$$\mathbf{e}(\lambda r/2) \lesssim \mathbf{f}(\lambda r) + \left(\frac{\varepsilon}{r}\right)^\theta \mathbf{e}(\lambda r) + Q_{1-\theta}\left(\frac{r}{\varepsilon}\right)$$

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$$Q_{1-\theta}(R) := \int_{\mathbb{R}^n \setminus B_{R^{1-\theta}}} |x| G(x) \, \mathrm{d}x$$

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6. Iterating, we get power decay of the excess.

Conclusion

- Minimizers have $C^{1,\alpha}$ boundary, with regularity constants not depending on ε
- For small ε, any minimizer is nearly spherical

Theorem (M. Goldman, B. Merlet, M.P. – Minimality of the ball)

For every $n \geq 2$ and every $0 < \gamma < 1$, there exists $\varepsilon_* > 0$ s.t. the unit ball is the unique minimizer of ($\star$), up to translations, as long as $\varepsilon < \varepsilon_*$.

Thank you!

u is almost harmonic (1)

- In the ball B_r, we consider variations $f_t(x) = x + tT(x)$ where $T(x) = \varphi(x')e_n$, $\varphi \in C_c^\infty(D_r)$, and competitors $E_t = f_t(E_\varepsilon)$

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- "Localized" Euler–Lagrange

$$\int_{\partial^* E_\varepsilon \cap B_{2r}} \nu_{E_\varepsilon} \cdot (\nabla T \nu_{E_\varepsilon}) \qquad + \text{"smaller terms"}$$
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- Rewrites

$$\int_{\mathbb{R}^{n-1}} \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1+|\nabla u|^2}} \qquad + \text{“smaller terms”}$$

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that is, $(\Delta - \gamma \Delta_{G_\varepsilon}) u \simeq 0$ in B_r.

Caccioppoli inequality

- Setting $\mathscr{E}(E,t) = P(E; D_t \times (-1,1)) - \mathcal{H}^{n-1}(D_t)$
  (imagine $\mathscr{E}(E_\varepsilon, t) \simeq \mathbf{e}(E_\varepsilon, B_t)$),
  by minimality we show

  $$\mathscr{E}(E_\varepsilon, t) \lesssim C\mathscr{E}(F,t) + C[\mathscr{E}(E_\varepsilon, t+\varepsilon^\theta) - \mathscr{E}(E_\varepsilon, t)] + Q_{1-\theta}(1/\varepsilon) \simeq 0,$$

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- By a covering argument, we deduce (strong Caccioppoli)

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