Γ-Convergence of the Ginzburg-Landau Functional with tangential and normal boundary conditions

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We consider the Ginzburg-Landau functional, E_ε, in a problem with tangential $u \cdot \nu = 0$ (or normal $u \cdot \tau = 0$) boundary conditions. These are motivated by experimental results of Volovik and Lavrentovich [4] in which nematic drops are placed in an isotropic medium, allowing for the control of nematic boundary behaviour. For 3D samples, Volovik and Lavrentovich found a single interior hedgehog defect when molecules are asked to be normal to the boundary and a bipolar boojum pair when requiring tangential conditions. Inspired by the physical phenomena observed in [4], in [1] the first two authors studied minimizers for the Ginzburg-Landau energy in a 2D setting with tangential and normal boundary conditions. In particular, they show that minimizers can exhibit half-degree vortices on the boundary. We study the full Γ-convergence in the setting from [4], by extending the work of Jerrard and Soner from [3] (see also [2] for a related problem). In their work, Jerrard and Soner relate, through the framework of Γ-convergence, convergence of the Jacobian in the interior of the domain $\Omega \subseteq \mathbb{R}^2$ to lower bounds on energy, and hence to the formation of interior defects. We show that, under appropriate restrictions of the functions along the boundary, we may extend the convergence of the Jacobian to hold up to the boundary and hence recover boundary defects as well.

Théorème 1. *1. If $\{u_\varepsilon\}_{\varepsilon\in(0,1]} \subseteq W_T^{1,2}(\Omega;\mathbb{R}^2)$ satisfies $E_\varepsilon(u_\varepsilon) \leq C|\log(\varepsilon)|$ for all $\varepsilon \in (0,1]$ and some $C > 0$. then, up to a subsequence that we do not relabel, we have that there is a signed Radon measure, $J_* = \pi \sum_{i=1}^{M_1} d_i\delta_{a_i} + \frac{\pi}{2}\sum_{j=0}^{b}\sum_{k=1}^{M_{2,j}} d_{jk}\delta_{c_{jk}}$, where $d_i, d_{jk} \in \mathbb{Z}\setminus\{0\}$, $a_i \in \Omega$, and $c_{jk} \in (\partial\Omega)_j$ $\forall i,j,k$, satisfying*

$$\lim_{\varepsilon\to 0^+} \|\star J(u_\varepsilon) - J_*\|_{(C^{0,\alpha}(\Omega))^*} = 0, \qquad |J_*| \leq \liminf_{\varepsilon\to 0^+} \frac{E_\varepsilon(u_\varepsilon)}{|\log(\varepsilon)|}, \tag{1}$$

$$\sum_{i=1}^{M_1} d_i + \frac{1}{2}\sum_{j=0}^{b}\sum_{k=1}^{M_{2,j}} d_{jk} = \chi_{Euler}(\Omega), \qquad \frac{1}{2}\sum_{k=1}^{M_{2,j}} d_{jk} \in \mathbb{Z} \text{ for each } j = 0,1,\ldots,b. \tag{2}$$

2. For each such signed measure J_* satisfying (2) we can find $\{u_\varepsilon\}_{\varepsilon\in(0,1]} \subseteq W_T^{1,2}(\Omega;\mathbb{R}^2)$ such that

$$\lim_{\varepsilon\to 0^+} \|\star J(u_\varepsilon) - J_*\|_{(C^{0,\alpha}(\Omega))^*} = 0, \;\; \forall\, 0 < \alpha \leq 1, \qquad \limsup_{\varepsilon\to 0^+} \frac{E_\varepsilon(u_\varepsilon)}{|\log(\varepsilon)|} = \|J_*\|.$$

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