SMAI MODE 2023 22-26 mai 2023

# Parameters Estimation of the Ding and al. Model to Optimize the Muscular Response to FES to Design a Smart Electrostimulator

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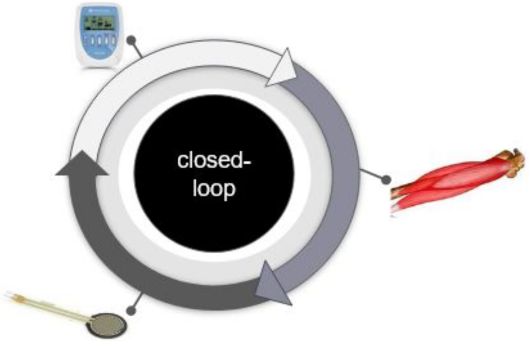


# Mathematical Equations

The input $u$ of a pulse train is defined, for $t \in$ [0, tf] y:

$$u(t) = \sum_{i=0}^{n} \delta(t - t_i) \quad (1)$$

with $0 = t_0 < t_1 < \cdots < t_n <$ tf impulse times with $n \in \mathbb{N}$ fixed.

The dynamics is given by:

$$\dot{E}(t) + \frac{E(t)}{\tau_c} = \frac{1}{\tau_c} \sum_{i=0}^{n} \eta_i R_i \delta(t - t_i), \quad (2)$$

with $E(0) = 0$, $\eta_i$ is the amplitude of the electric pulse stimulation, $\tau_c$ the response time and the function $R_i$ defined by $R_i = 1$ if $i = 0$ and $R_i = 1 + (R_0 - 1)e^{-\frac{t_i - t_{i-1}}{\tau_c}}$ elsewhere .

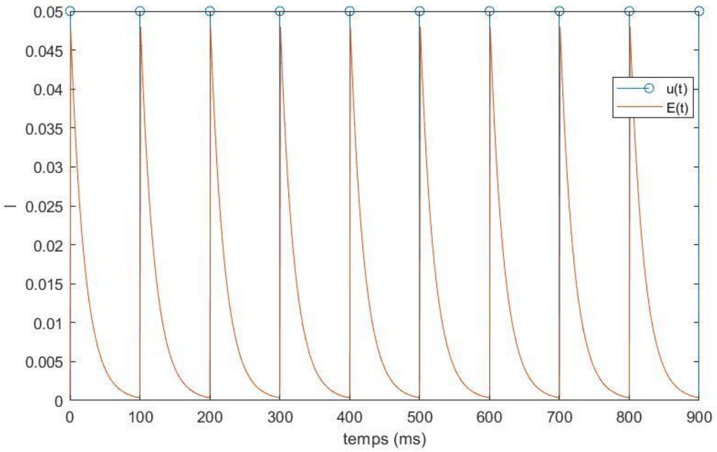



fig : E(t), T = 100ms

$$\dot{c}_N(t) + \frac{c_N(t)}{\tau_c} = E(t), \qquad (3)$$

The dynamics is given by:

$$m_1(t) = \frac{c_N(t)}{K_m + c_N(t)}, \quad m_2(t) = \frac{1}{\tau_1 + \tau_2\, m_1(t)}, \qquad (4)$$

with $m_1$ the *Michaelis-Menten* (1913) function. The muscular force response satisfied the Hill-Huxley's models dynamics.

$$\dot{F}(t) = -m_2(t)\, F(t) + m_1(t) A, \qquad (5)$$

with $\tau_c, R_0, A_0, K_m, \tau_1, \tau_2$ the set of parameters

$$F(t) = A_0 M(t) \int_0^t M^{-1}(s) m_1(s)\, ds, \qquad (6)$$

with $M(t) = \exp\left( - \int_0^t m_2(s)\, ds \right)$.

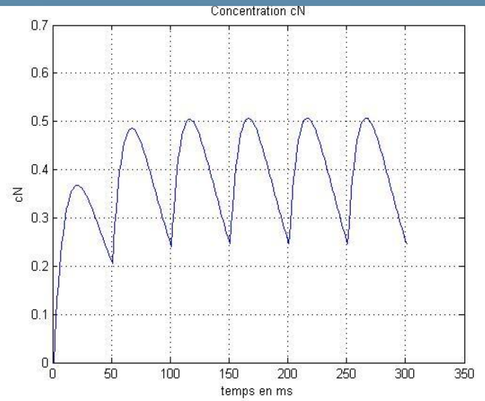



fig : cN(t), T = 50ms

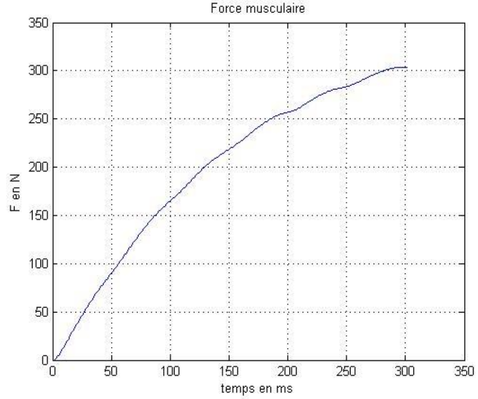



fig : F(t), T = 50ms

Construction of the approximation of the muscular force $F$ on $[t_{k_t+j_t/p}, t_{k_t+(j_t+1)/p}]$, $k_t = 0, \ldots, n$, $j_t = 0, \ldots, p-1$ is given by :

$$\tilde{F}(t) = A_0 M(t) \int_0^t \tilde{M}^{-1}(s) \tilde{m}_1(s)\, ds, \qquad (7)$$

with $\tilde{M}(t) = \exp\left(-\int_0^t \tilde{m}_2(s)\, ds\right)$, et $\tilde{m}_i$, $i = 1, 2$ an approximation of $m_i$ defined by :

$$\tilde{m}_i(t) = a_{ij,k}\,(t - t_{k+j/p}) + b_{ij,k}, \quad \text{for } t \in [t_{k+j/p}, t_{k+(j+1)/p}]$$

and $k = 0, \ldots, n$, $j = 0, \ldots, p-1$. The constants $a_{ij,k}, b_{ij,k}$, $j = 0, \ldots, p-1$ are calculed at specific times, depending on the choice on the approximation of $m_i$.

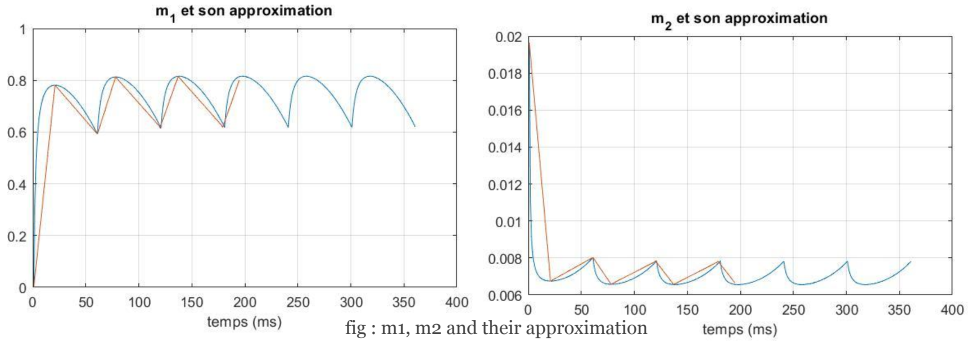



fig : m1, m2 and their approximation

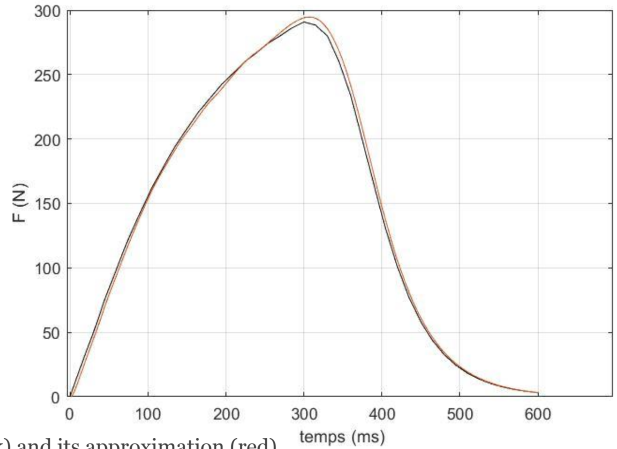



fig : F (black) and its approximation (red)

The approximation of the muscular force $F$ on $[t_{k_t+j_t/p}, t_{k_t+(j_t+1)/p}]$, $k_t = 0,\dots,n$, $j_t = 0,\dots,p-1$ is given by :

$$\tilde{F}(t)/A_0 = \sum_{i=0}^{k_t-1}\sum_{j=0}^{p-1}\int_{t_{i+j/p}}^{t_{i+(j+1)/p}} \tilde{M}(t)\tilde{M}^{-1}(s)\,\tilde{m}_1(s)\,ds + \sum_{j=0}^{j_t-1}\int_{t_{k_t+j/p}}^{t_{k_t+(j+1)/p}} \tilde{M}(t)\tilde{M}^{-1}(s)\,\tilde{m}_1(s)\,ds + \int_{t_{k_t+j_t/p}}^{t} \tilde{M}(t)\tilde{M}^{-1}(s)\,\tilde{m}_1(s)\,ds. \quad (8)$$

From a practical point of view, we just need to estimate the values of the force $F$ for $t_{k+j/p}$, which gives :

$$\tilde{F}(t_{k+j/p})/A_0 = \sum_{i=0}^{k-1}\sum_{j=0}^{p-1}\int_{t_{i+j/p}}^{t_{i+(j+1)/p}} \tilde{M}(t_{k+j/p})\tilde{M}^{-1}(s)\,\tilde{m}_1(s)\,ds + \sum_{j=0}^{j-1}\int_{t_{k+j/p}}^{t_{k+(j+1)/p}} \tilde{M}(t_{k+j/p})\tilde{M}^{-1}(s)\,\tilde{m}_1(s)\,ds \quad (9)$$

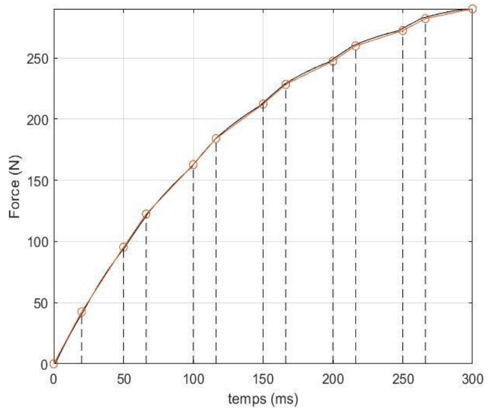
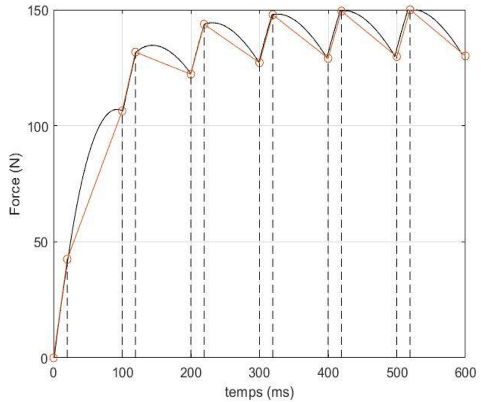
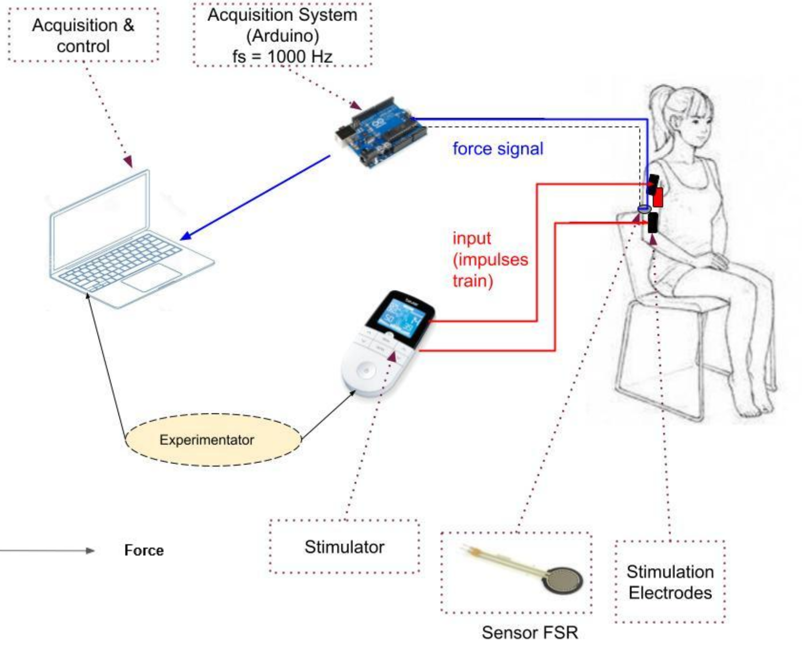
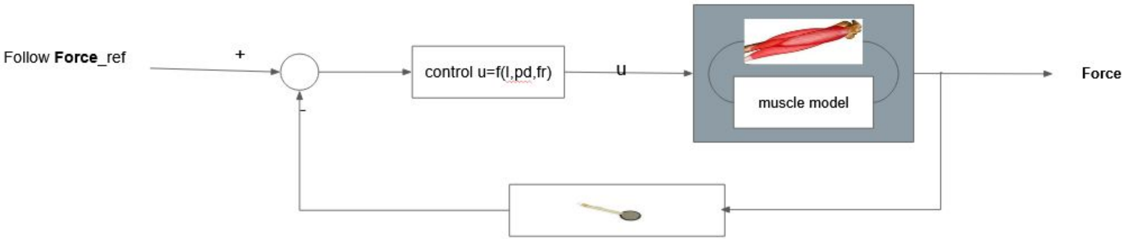


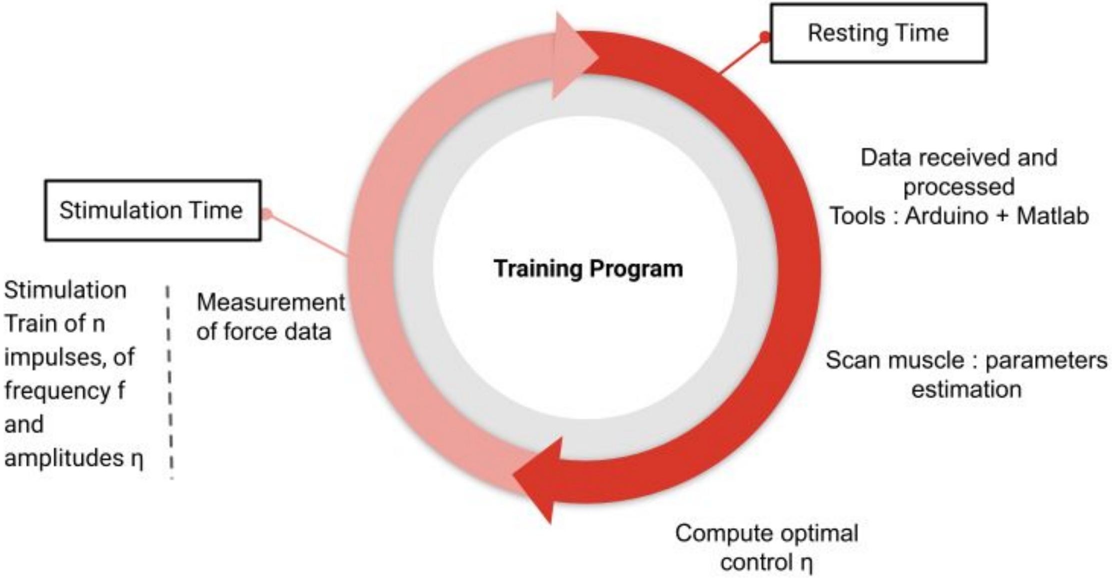

fig : (8) illustrated in black, (9) illustrated in red



- 8 individuals
- isometric conditions
- 1 Hz, 2 Hz, 5 Hz, 10 Hz




Resting Time
Stimulation Time
Training Program
Data received and processed
Tools : Arduino + Matlab
Scan muscle : parameters estimation
Compute optimal control η
Stimulation Train of n impulses, of frequency f and amplitudes η
Measurement of force data

**Model** (9) gives the approximation of the force in terms of a set $\alpha$ of 6 parameters and $t_{k+j/p}$ with $k = 0, \dots, n$ ($n$ number of pulses during the train), $j = 0, \dots, p-1$ and $p \in \mathbb{N}^*$ the number of the new points added between $t_k$ and $t_{k+1}$.
$\alpha = [\tau_c, R_0, A_0, K_m, \tau_1, \tau_2]$.

**Data** For each bounce $k$, the average value is calculated using the data value $F_{data}$:

$$\bar{F}_{[t_k,t_{k+1}]} = \frac{1}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} F_{data}(t)\, dt$$

The vector of data force, is defined for each impulse $k \in [1, n-1]$ by $F_{data-fit} = [F_{data}(t_k), \bar{F}_{[t_k,t_{k+1}]}]$

$$F_{data-fit} = [F_{data}(t_1), \bar{F}_{[t_1,t_2]}, \dots, F_{data}(t_{n-1}), \bar{F}_{[t_{n-1},t_n]}]$$

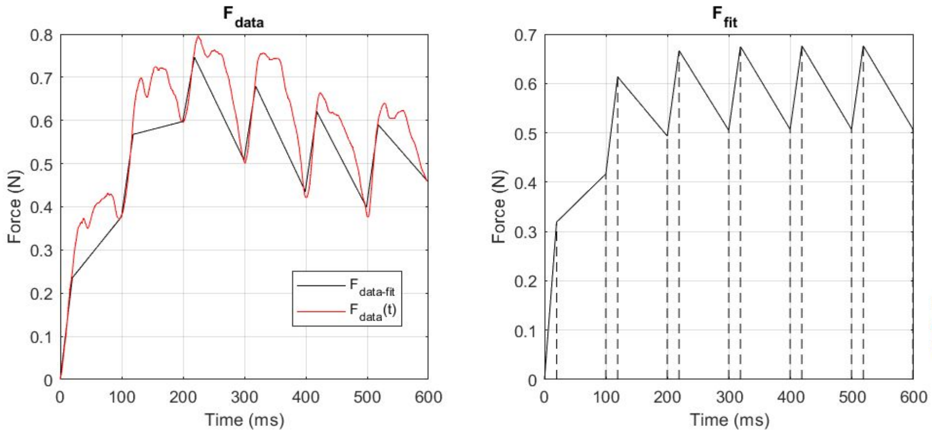
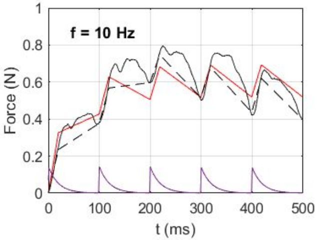


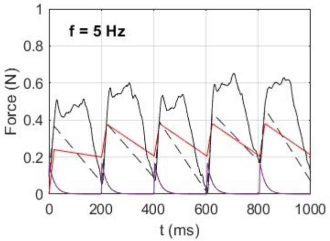

Fig. At the right, $F_{fit}$ with the dashed curves corresponding to the times $t_{k+j/p}$. At the left, $F_{data}(t)$ and $F_{data-fit}$ the vector of values which contains the data that will be fitted to the model. $p = 2$ and $n = 6$.

Each value of $F_{data-fit}$ fits the model $F(t_{k+j/p})$ which corresponds to the approximation force calculated for each impulse time $t_k$ and for $t_{k+j/p}$, the intermediate times between 2 pulses.
For $k \in [1, n]$, the values $F_{data}(t_k)$ fit $\tilde{F}(t_k)$ and the values $\bar{F}_{[t_k, t_{k+1}]}$ fit $\tilde{F}(t_{k+j/p})$ ($p = 2$ and $j = 1$). The cost function to minimize is described below:

$$J(\alpha) = \sum_{i=1}^{N} \|\tilde{F}_i(t_{k+j/p}, \alpha) - F_{i,data-fit}\|_2^2$$

$\alpha = [\tau_c, R_0, A_0, K_m, \tau_1, \tau_2]$. $N$ the size of the data and the fit vectors ($N = n.p$).



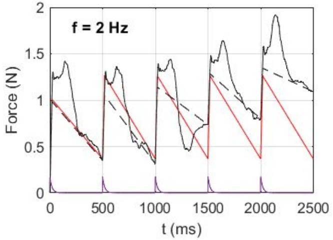




Fig. 5. The black curve corresponds to the force data measured with the FSR sensor. The dashed curve is the average force data measured at $t_{k+j/p}$ with $k = 0, \ldots, n$, $j = 0, \ldots, p-1$ and $p \in \mathbb{N}^*$ the number of the new points added between $t_k$ and $t_{k+1}$. The red curve represents the approximation of the force that fits the measured data. Values of the constants are $n = 5$ and $p = 2$. The computation time is less than 2s.

The resulting error (%) between the measured data $F_{data-fit}$ and the estimated model response $F_{fit}$ is defined as follows:

$$E(\%) = \frac{\sum_i (F_{data-fit} - F_{fit})^2}{\sum_i (F_{data-fit})^2} \times 100$$

Table 1: Modeling errors calculated for the different subjects, according to the stimulation frequencies.

| E (%) | 2Hz | 5Hz | 10Hz |
|---|---|---|---|
| subjet 1 | 9,50 | 12,89 | 17,89 |
| subjet 2 | 11,06 | 26,87 | 13,39 |
| subjet 3 | 2,36 | 11,49 | 30,57 |
| subjet 6 | 32,35 | 20,89 | / |
| subjet 8 | 12,9 | 8,61 | 10,41 |

Table 2: Estimated experimental parameters for one individual biceps for different stimulation frequency.

| $\alpha$ | $\tau_c$ (ms) | $R_0$ | $K_m$ | $A_0$ (N/ms) | $\tau_1$ (ms) | $\tau_2$ (ms) |
|---|---|---|---|---|---|---|
| 1Hz | 19.83 | 1.14 | 4.38 | 1.31 | 50.91 | 124.39 |
| 2Hz | 19.86 | 1.14 | 4.82 | 1.14 | 50.88 | 124.40 |
| 5Hz | 19.87 | 1.13 | 4.87 | 0.77 | 50.83 | 124.39 |
| $\bar{\alpha}$ | 19.82 | 1.13 | 4.69 | 0.94 | 50.86 | 124.39 |
| $\alpha_{Ding}$ | 20 | 1.14 | 0.1030 | 3.009 | 50.95 | 124.40 |

We consider the non-linear system:

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \quad (10)$$

with $x \in \mathbb{R}$ and $y \in \mathbb{R}$, the single output.

**Universal or admissible input**

An input $u$ is said to be universal if it distinguishes any pair of distinct points $(x_1, x_2)$ on $[0,T]$ (i.e., $\exists t \in [t_0, t_0+T]$) such that $y(t,x_1,t_0,u) \neq y(t,x_2,t_0,u)$. Otherwise $u$ is a singular input.

**Rank condition**

The nonlinear system (10) is observable in the sense of rank if rank $\frac{\partial}{\partial x}\begin{bmatrix} h(x) \\ L_f h(x) \\ \dots \\ L_f^{n-1} h(x) \end{bmatrix}$ = n, with $L_f^j h(x)$ the successive Lie derivatives of the output $h(x)$ and $L_f^0 h(x) = h(x)$.

**Uniform observability**

The system (10) is said to be uniformly observable if, for all $T > 0$, any input $u : [0,T] \mapsto U$ is universal. In the case where the set of inputs $U$ is not universal, we can define a smaller set of universal inputs $\tilde{U} \subset U$, we say that the system is locally observable on $\tilde{U}$.

## Model

- inputs $u$ are bounded in $[\beta_0, \beta_1]$ small
- change of scale $\eta \Rightarrow \eta\epsilon$ (where $\eta$ describes the amplitude of the stimulation and $\epsilon$ is a very small quantity)
- $u(t) = c_N(t)\epsilon$

$$f = A_0 m1(t) - F(t)m_2(t) = A_0 \frac{c_N(t)\epsilon}{K_{m_0} + c_n(t)\epsilon} - F(t)\frac{1}{\tau_1 + \tau_2 \frac{c_n(t)\epsilon}{K_m + c_n(t)\epsilon}}$$

Using Taylor series with a first order, we have:

$$f(F(t), c_N(t)) = \frac{A_0 c_N(t)}{K_{m_0}}\epsilon - \frac{F(t)}{\tau_1} + \frac{c_N(t)F(t)\tau_2}{K_{m_0}\tau_1^2}\epsilon + o(\epsilon^2)$$

The equations of (10) are:

$$f(x,u) = \begin{cases} x_2 u(t) - x_1\left(\frac{1}{\tau_1} - \frac{\tau_2 u(t)}{x_3 \tau_1^2}\right) \\ 0 \\ 0 \end{cases} \quad (11)$$

with $h(x) = x_1$, $x = [F, \frac{A_0}{K_{m_0}}, K_{m_0}] \in \mathbb{R}^3$ , $y \in \mathbb{R}$. $f$ and $h$ are $C^\infty$. The inputs $u(t) = a + b\cos(2\pi f t)$ are bounded for $t$ in $[t_i, t_{i+1}[$, $i = [1, n]$, $n$ number of pulses in a train. This form modelizes the dynamics of (3) with the bounces. The functions $f$ and $h$ are $C^\infty$.

We introduce the change of variable:

$$\Phi = \begin{cases} \Phi : \mathbb{R}^3 \mapsto \mathbb{R}^3 \\ x \mapsto \Phi(x) = [h(x), L_f h(x), L_f^2 h(x)] \end{cases} \qquad (12)$$

with

$$\begin{cases} h(x(t)) = x_1 \\ L_f h(x(t)) = x_2 u(t) - x_1 (c - \frac{du(t)}{x_3}) \\ L_f^2 h(x(t))] = x_2 \dot{u}(t) + L_f h(x(t))(\frac{du(t)}{x_3} - c) + \frac{x_1 d\dot{u}(t)}{x_3} \end{cases} \qquad (13)$$

with $c = \frac{1}{\tau_1}$ et $d = \frac{\tau_2}{\tau_1^2}$. The system (10) is written in the new coordinates $z = \Phi(x)$, in a observability canonical form. We say that $\Phi$ is a diffeomorphism.

$$\begin{cases} \dot{z} = Az(t) + \rho(z(t, u(t)) \\ y(t) = h(z) \end{cases} \qquad (14)$$

with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $C = [1, 0, 0]$.

and $\rho(z(t), u(t)) = \begin{bmatrix} 0 \\ 0 \\ L_f^3 h(x(t)) \end{bmatrix}$ is bounded with respect to $z$ and $u \in \tilde{U}$. And $L_f^3 h(x(t))$ is a function of $u$, $\dot{u}$ and $\dot{x}_2$.

**Theorem**

*If (A,C) is observable and is under the previous conditions, then the system (14) admits an observer of the form :*
$\dot{\hat{z}} = A\hat{z} + \rho(\hat{z}, u) + S^{-1} K_0 (y(t) - \hat{z}(t))$

where:

- $\Theta \in \mathbb{R}$ is an observer setting parameter which allows to play on the speed of convergence of the observer.
- $S$ is a symmetric positive definite matrix given by the Lyapunov equation $\Theta S + A^T S + SA = C^T C$.
- $K_0$ is defined such that $(A - K_0 C)$ is Hurwitz.

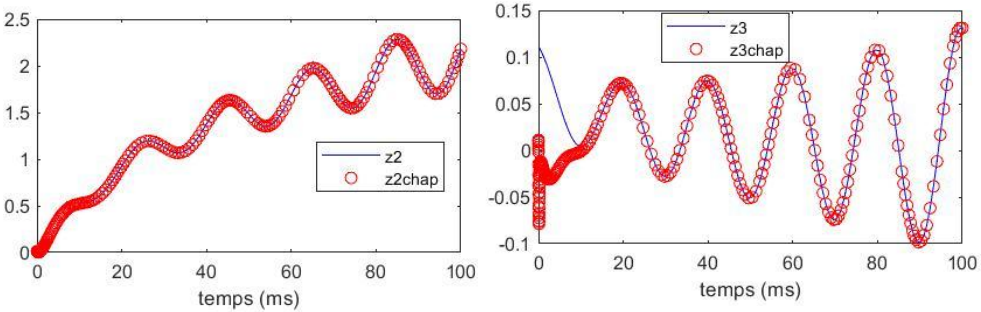



Fig. Estimation $\hat{z}$ for $t \in [0, 100]$ from $x_0 = [F_0 \frac{A(0)}{K_m(0)}, K_m(0)] = [0.001, 30, 0.10]$, $\hat{x_0}(t) = \Phi^{-1}(\hat{z_0}(t))$ and $\hat{z}_0 = [0.1 z_0(1), 1.001 z_0(2), 0.1 z_0(3)]$. $K_0 = [15, 75, 125]$, $\Theta = 5$.

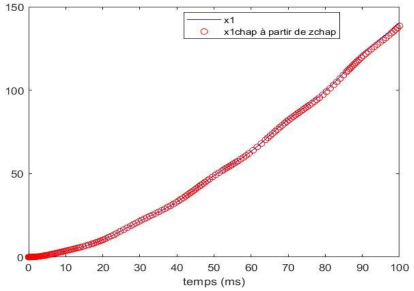
Once the state vector $z(t)$ is estimated, we can go back to the estimate of $x(t)$ by $\hat{x}(t) = \Phi^{-1}(\hat{z}(t))$.
We have:

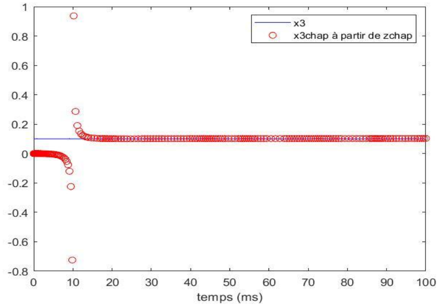
$$\begin{cases} x_1 = z_1 \\ x_2 = \frac{u\dot{u}z_1^2 + \dot{u}z_1z_2 - uz_3z_1 + uz_2^2}{u^2z_2} \\ x_3 = -\frac{d\dot{u}^2z_2}{\dot{u}z_2 - uz_3 + c\dot{u}z_1 - cuz_2} \end{cases}$$

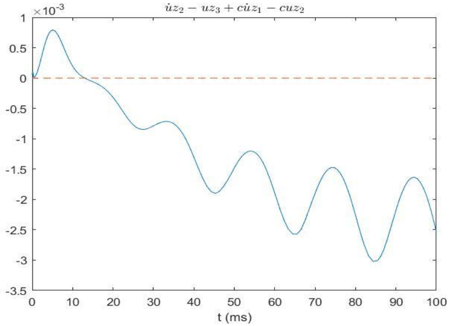
Singular inputs : They are defined by:

$$u^2z_2 \neq 0 \Rightarrow \begin{cases} u \neq 0 \\ z_2 \neq 0 \end{cases}$$

$$\dot{u}z_2 - uz_3 + c\dot{u}z_1 - cuz_2 \neq 0$$



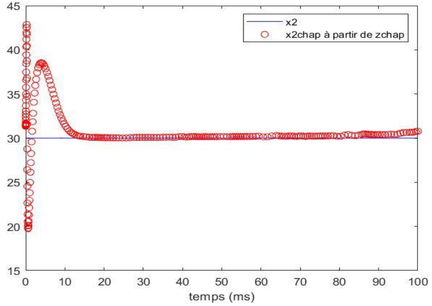




Fig. Estimation $\hat{x}$ for $t \in [0,100]$ from $x_0 = [F_0\frac{A(0)}{K_m(0)}, K_m(0)] = [0.001, 30, 0.10]$, $\hat{x}_0(t) = \Phi^{-1}(\hat{z}_0(t))$ and $\hat{z}_0 = [0.1z_0(1), 1.001z_0(2), 0.1z_0(3)]$. $K_0 = [15, 75, 125], \Theta = 5$.

*a) Endurance program:* We consider a single train $[0,T]$ on which the corresponding problem is:

$$\min_{x} \int_0^T |F(t) - F_{ref}|^2 \, dt \qquad (17)$$

The amplitudes are optimization variables ($x = (\eta_1, \eta_2, ..., \eta_n)$), which belong to the finite dimensional input-space $I \in [0,1]^{n+1}$ defined by amplitude constraints: $\eta_i \in [0,1]$, $i = 0, \ ... \ , n$, with $n$ a fixed integer, representing the number of pulses in the train. $T$ is the time of the end of the pulse train. The reference force $F_{ref}$ is adjusted in relation with the user.

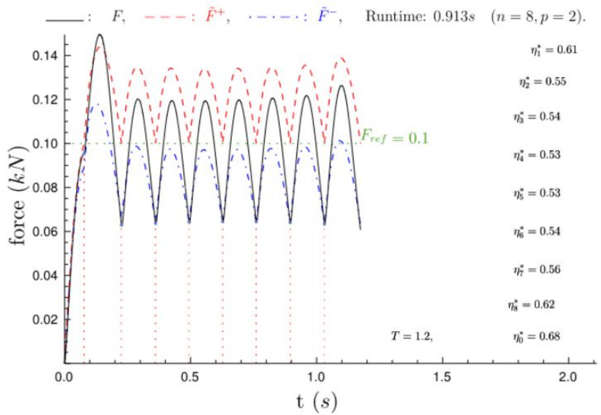



Fig. The optimal solution $(\eta_0^*, ..., \eta_n^*)$ minimizes the cost $\tilde{J}_1$ from (19). $\tilde{F}^+$ and $\tilde{F}^-$ are respectively the upper and the lower approximation of F, described in the article [5]. $F_{ref} = 0.1$ kN.

$$\min_{\eta_i} \tilde{J}_1 = \min_{\eta_i} \sum_{k=1}^{n} \|\tilde{F}_k(t_{k+j/p}, \alpha, \eta_i) - F_{ref}\|_2^2 \qquad (19)$$

with $\alpha = [\tau_c, R_0, A_0, K_m, \tau_1, \tau_2]$.

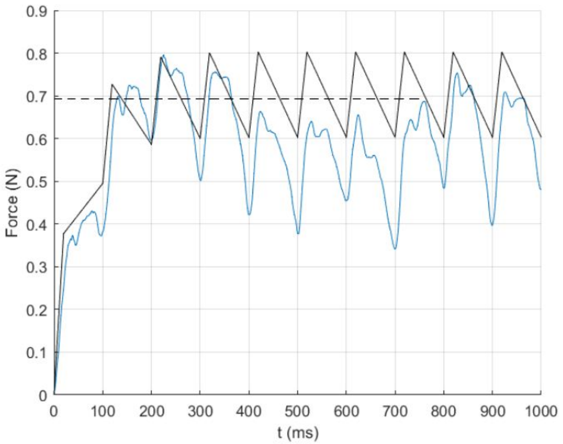



Fig. The optimal solution $(\eta_0^*, ..., \eta_n^*) = 0.6615$ minimizes the cost $J$ from (19). The black curve is the approximation force $\tilde{F}$, the blue curve is $F_{mesured}$ the data force measured with FSR sensor. The dashed curve is $F_{ref}$. Values of the constants are $\tau_c = 19.74$ms, $K_m = 6.86$, $R_0 = 1.10$, $A_0 = 0.68$, $\tau_1 = 50.84$ms, $\tau_2 = 124.39$ms, $n = 10$, $F_{ref} = 0.7$ N, the impulse frequency is constant and $f = 10$ Hz. Computation time is less than 3s.

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