Infering varifold structure from the data

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Context

Manifold estimation has been deeply investigated in the C^k-regularity model in [1]. However, data do not always lie on such a regular shape, our goal is to study geometric estimators under weaker hypothesis from the geometric measure theory standpoint [2]. For this purpose we use the varifolds framework, a weak version of manifold born in the 60's that is well adapted to surface approximation and that we now intend to analyse from a statistical perspective. Our approach take root in the seminal work [4].

Question

Given:

- a d- dimensional object S example: 2-sphere in $\mathbb{R}^3$
- a set of points $\{x_i\}_{i=1\ldots N}$ sampling S example: a scan acquisition of S

Problem:How to attribute weights $\{m_i\}_{i=1\ldots N}$ so as to correct sampling irregularities:we want that $\sum_{i=1}^N m_i\delta_{x_i}$ is close to the superficial measure $\mathcal{H}^d_{|S}$.

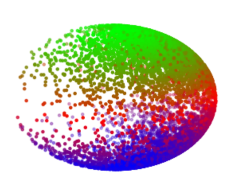

Fig. 1: Non uniformly sampled sphere

Statistical framework

We model the non uniformity of the sampling by a density θ: we assume that we have access to i.i.d samples $X_1,\ldots,X_N$ following the same law $\mu = \theta\mathcal{H}^d_{|S}$.

Definition 1 (Empirical measure). *Let $N\in\mathbb{N}^*$, $X_1,\ldots,X_N$ be N random independent variables with the same law μ, the associated empirical measure μ_N is defined as:*

$$\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$$

Question: how to recover θ and then $\mathcal{H}^d_{|S}$ from the knowledge of μ_N?

A proper distance

Definition 2 (Bounded Lipschitz distance). *For μ,ν two Radon measures in E (E can be $\mathbb{R}^n$ or $\mathbb{R}^n\times \mathrm{Sym}_+(n)$) , the bounded Lipschitz distance is defined by:*

$$\beta(\mu,\nu) = \sup\left\{\left|\int_E fd(\mu-\nu)\right| : f\in C_c(E,\mathbb{R}), ||f||_\infty\le 1, Lip(f)\le 1,\right\}$$

Lipschitz functions link the variation in the image spaces to the variation in E. It gives rougher convergence:

$$\beta(\delta_{\frac{1}{k+1}},\delta_0)\le Lip(f)\frac{1}{k+1}\xrightarrow[k\to+\infty]{} 0$$

comparatively to the total variation distance

$$d_{TV}(\delta_{\frac{1}{k+1}},\delta_0) = \sup\left\{\left|\int_{\mathbb{R}} fd(\delta_{\frac{1}{k+1}}-\delta_0)\right| : f\in C_c(\mathbb{R},\mathbb{R}), ||f||_\infty\le 1\right\} = 2.$$

Ahlfors regular measure

Definition 3 (Ahlfors regular measure). *Let μ be a Radon measure in $\mathbb{R}^n$, for $d>0$, we say that μ is a d-Ahlfors regular measure if $\exists C_0>0$ such that $\forall x\in supp(\mu)=S$, $\forall r\in]0,diam(S)]$:*

$$\frac{1}{C_0}r^d\le\mu(B(x,r))\le C_0r^d.$$

Examples: sphere, square, Koch snowflake,four-corner Cantor,...

Rectifiability

Definition 4 (d-rectifiable set). *A set S is d-rectifiable if there exists a countable family $(f_i)_{i\in\mathbb{N}}$ of Lipschitz maps from $\mathbb{R}^d$ to $\mathbb{R}^n$ such that*

$$\mathcal{H}^d\left(S\setminus\bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^d)\right)=0$$

Definition 5 (Rectifiable measures). *Let μ be a Radon measure in $\mathbb{R}^n$.We say that μ is d-rectifiable if there exist a d-rectifiable set S and a Borel function $\theta: S\mapsto\mathbb{R}_+$ such that $\mu=\theta\mathcal{H}^d_{|S}$.*

Now we want to analyse the local behaviour of our Radon measure μ around $x\in\mathbb{R}^n$, we use the rescaled measures for $r>0$:

$$\mu_{x,r}(B) := \mu(x+rB)\qquad \text{for } B\in Bor(\mathbb{R}^n)$$

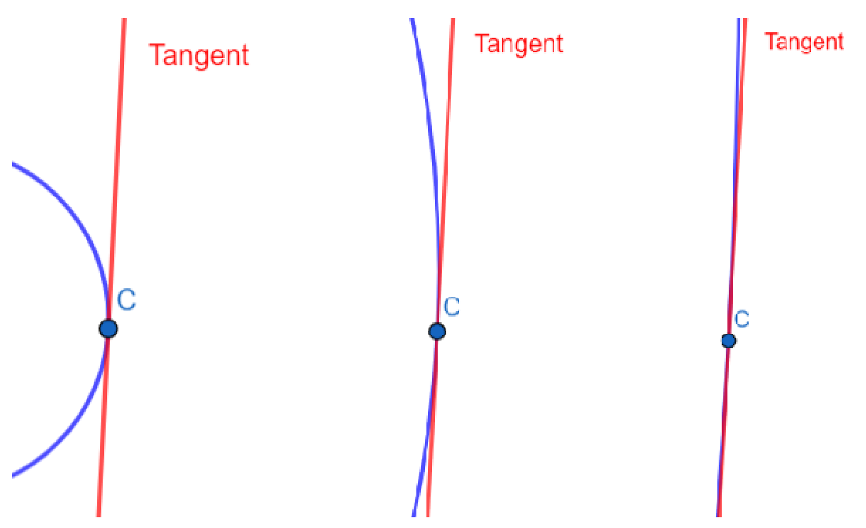

Fig. 2: Zoom the map to understand the local behaviour

Approximate tangent plane

Definition 6 (Approximate tangent plane). *Let μ be a Radon measure and let $x\in\mathbb{R}^n$. We say that μ has approximate tangent space $P\in G_{d,n}$ with multiplicity $\theta\in\mathbb{R}_+$ at x, if $r^{-d}\mu_{x,r}$ weak-$*$ converges to $\theta\mathcal{H}^d_{|P}$ in $\mathbb{R}^n$ as $r\to 0_+$. That is:*

$$\lim_{r\to 0_+} r^{-d}\int_{\mathbb{R}^n}\phi\left(\frac{y-x}{r}\right)d\mu(y) = \theta\int_P\phi(y)d\mathcal{H}^d(y)\qquad\forall\phi\in C_c(\mathbb{R}^n).$$

Proposition 1. *If $\mu=\theta\mathcal{H}^d_{|S}$ is d-rectifiable then μ admits an approximate tangent plan for $\mathcal{H}^d$ almost every $x\in S$*

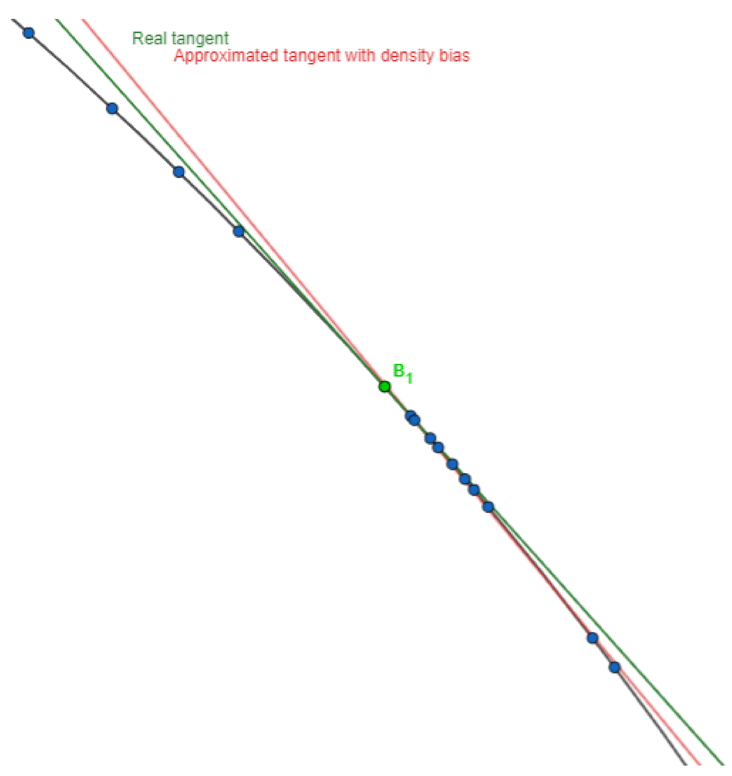

Fig. 3: Effect of the density on the tangent choice

Density Estimators

The framework is the following:

- $S\subset\mathbb{R}^n$ d-dimensional set: $\mathcal{H}^d(S)<+\infty$
- $\mu=\theta\mathcal{H}^d_{|S}$ d-Ahlfors probability measure
- $N\in\mathbb{N}$ and μ_N the empirical measure
- choose $\eta:\mathbb{R}\to\mathbb{R}_+$ satisfying $supp(\eta)\subset[-1,1]$ and $||\eta||_\infty + Lip(\eta)\le 1$

For $\delta>0$ we introduce

$$\forall x\in\mathbb{R}^n,\quad \theta_{\delta,N}(x) := \frac{1}{c_\eta\delta^d}\int_{\mathbb{R}^n}\eta\left(\frac{|x-y|}{\delta}\right)d\mu_N(y)$$

and

$$\theta_\delta(x) := \frac{1}{c_\eta\delta^d}\int_{\mathbb{R}^n}\eta\left(\frac{|x-y|}{\delta}\right)d\mu(y)$$

Theorem 1. *Let $(\delta_N)_{N\in\mathbb{N}^*}$ be a positive sequence tending to 0 and such that $\delta_N N^{\frac{1}{d}}\xrightarrow[N\to+\infty]{}+\infty$, then for $\mathcal{H}^d$–a.e. $x\in S$,*

$$\mathbb{E}\left[\left|\theta_{\delta_N,N}(x)-\theta(x)\right|\right]\le C\frac{N^{-\frac{1}{d}}}{\delta_N}+|\theta_{\delta_N}(x)-\theta(x)|\xrightarrow[N\to+\infty]{}0\,.$$

Tangent plane estimators

Let Φ be a Lipschitz truncation of the inverse function, for $0<\tau\le 1$ and $t>0$,

$$\Phi(t)=\frac{\chi_\tau(t)}{t}\quad\text{and}\quad \chi_\tau(t)=\begin{cases}0 & \text{if } 0<t<\frac{\tau}{2}\\ \frac{2}{\tau}t-1 & \text{if } \frac{\tau}{2}\le t\le\tau\\ 1 & \text{if } t>\tau\end{cases}.$$

We introduce

$$\nu_{\delta,N} := \frac{1}{N}\sum_{i=1}^N\Phi(\theta_{\delta,N}(X_i))\delta_{X_i} = \left(\Phi\circ\theta_{\delta,N}\right)\mu_N\quad\text{and}\quad \nu_\delta = (\Phi\circ\theta_\delta)\,\mu$$

β_B: localized version of β with test functions supported in B

Definition 7 (Covariance matrix associated with a measure). *Let $0<d\le n$, $r>0$ and let λ be a Radon mesure in $\mathbb{R}^n$. We define for $x\in\mathbb{R}^n$, the $n\times n$ matrix*

$$\Sigma_r(x,\lambda) = \frac{1}{\sigma_\phi r^d}\int_{\mathbb{R}^n}\phi\left(\frac{|y-x|}{r}\right)\frac{y-x}{r}\otimes\frac{y-x}{r}\,d\lambda(y)\,,$$

where $z\otimes z$ is the rank one matrix of (i,j)–coefficient z_iz_j, for $z\in\mathbb{R}^n$ and $\sigma_\phi=\omega_d\int_{r=0}^1\phi(r)r^{d+1}\,dr$.

Proposition 2. *Let $\mu=\theta\mathcal{H}^d_{|S}$ be a d–rectifiable measure, then for $\mathcal{H}^d$–a. e. $x\in S$,*

$$\Sigma_r(x,\mu)\xrightarrow[r\to 0_+]{}\theta(x)\Pi_{T_xS}$$

where Π_{T_xS} is the matrix of orthogonal projection on the approximate tangent space T_xS.

Varifold estimators

Definition 8. *Let $W_{r,\delta,N}:=\nu_{\delta,N}\otimes\delta_{\Sigma_r(x,\nu_{\delta,N})}$ and $W_{r,\delta}:=\nu_\delta\otimes\delta_{\Sigma_r(x,\nu_\delta)}$, our Radon measures on $\mathbb{R}^n\times\mathrm{Sym}_+(n)$ that will converge to the varifold structure behind S.*

Theorem 2. *Assume that S is d-rectifiable, $0<\theta_{\min}\le\theta\le\theta_{\max}$ and μ is d-Ahlfors regular with C_0 the regularity constant of μ. . Then there exists a constant $M'=M'(d,C_0,\eta,\phi)>0$ such that for all open ball $B\subset\mathbb{R}^n$ of radius $R_B<1$, for all $r,\delta>0$ and $N\in\mathbb{N}^*$ satisfying $N^{-\frac{1}{d}}<R_B$ and $N^{-\frac{1}{d}}<\min(\delta,r)$,*

$$\mathbb{E}\left[\beta_B(W_{r,\delta,N},W_{r,\delta})\right]\le\begin{cases}\dfrac{M'\mu(B)}{\tau^2\min(\delta,r)}N^{-\frac{1}{2}}\ln N & \text{if } d=2\\[2ex] \dfrac{M'\mu(B)}{\tau^2\min(\delta,r)}N^{-\frac{1}{d}} & \text{if } d>2\end{cases}.$$

References

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