DE LA RECHERCHE À L'INDUSTRIE

PolyMAC: A Comparison

Staggered Finite Volume Methods on General Meshes

SMAI, 25 mai 2023

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Outline

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- PolyMAC: **family** of numerical schemes for multiphase compressible flows on **general** meshes (hexahedra, prisms, tetrahedra,...)
- Currently, three methods implemented in TRUST platform $=>$ PolyMAC I, II and III.
- Simplification: incompressible NS equations.

Objectives

- Description of the schemes
- Comparison: strengths/weaknesses
- Define guidelines for the user

Find $\vec{u}$ and p such that

$$\begin{aligned} \partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} - \nu \Delta \vec{u} + \nabla p &= \vec{f} \quad \text{in } \Omega\,, \\ \nabla \cdot \vec{u} &= 0 \quad \text{in } \Omega\,, \end{aligned} \tag{1}$$

- $\vec{u}$: fluid velocity
- p: pressure
- $\nu > 0$: viscosity
- $\Omega \subset \mathbb{R}^d$: unit square in 2D (unit cube in 3D)

Outline

PolyMAC: Finite Volume (FV) **generalization** of the MAC scheme to polyhedral meshes. The main unknowns are then:

- normal component of the **velocity** at the **faces**
- **pressure** at the **cells**

Approach

1. Definition of a benchmark (2D and 3D)
2. Estimation of the order of convergence
3. Density of linear systems
4. Guidelines for optimal use

Reformulation of Equation (1) with the vorticity:

$$\begin{aligned} \partial_t \vec{u} + \nabla \cdot (\vec{u} \otimes \vec{u}) + \nu \nabla \times \vec{\omega} + \nabla p &= \vec{f} \quad \text{in } \Omega\,, \\ \nabla \times \vec{u} - \vec{\omega} &= 0 \quad \text{in } \Omega\,, \\ \nabla \cdot \vec{u} &= 0 \quad \text{in } \Omega\,. \end{aligned} \tag{2}$$

- Gradient: mimetic finite volumes (MFV)
- Similar to Beltman *et al.* (2018)
- Convection: velocity at the cells to approach $\nabla \cdot (\vec{u} \otimes \vec{u})$ and interpolation to the faces

Resulting linear system:

$$\begin{pmatrix} \frac{M_{\mathbf{u}}}{\Delta t} + C(\mathbf{u}_f^t) & R & G \\ R^T & -\frac{1}{\nu} M_\omega & 0 \\ G^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_f^{t+\Delta t} \\ \nu \boldsymbol{\omega}_e^{t+\Delta t} \\ \mathbf{p}_c^{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \frac{M_{\mathbf{u}}}{\Delta t} \mathbf{u}_f^t \\ 0 \\ 0 \end{pmatrix}, \quad (3)$$

- $M_{\mathbf{u}}$: complicated matrix
- $C(\mathbf{u}_f^t)$: convection $\Rightarrow$ NS linearised at each time step
- Without convection, symmetric matrix
- **Saddle-point** system

Saddle-point in Equation (3) $\Rightarrow$ Difficult to implement solution.

PolyMAC II: introduction of the **velocity vector** at the cells as auxilary unknowns $\Rightarrow$ Simplification of the top-left block.

- Discretization of the momentum equation at the cells
- Interpolate the convective and diffusive terms at the faces
- MPFA for the gradients (Aavatsmark 2002)

Resulting linear system:

$$\begin{pmatrix} \frac{I}{\Delta t} & C_f(\mathbf{u}_c^t) + D_f(\mathbf{u}_c^t) & G_f^{MPFA} \\ 0 & \frac{I}{\Delta t} + C_c(\mathbf{u}_c^t) + D_c(\mathbf{u}_c^t) & G_c^{MPFA} \\ G^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_f^{t+\Delta t} \\ \mathbf{u}_c^{t+\Delta t} \\ \mathbf{p}_c^{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \frac{I}{\Delta t}\mathbf{u}_f^t \\ \frac{I}{\Delta t}\mathbf{u}_c^t \\ 0 \end{pmatrix} \tag{4}$$

PolyMAC II solves the saddle-point problem, but very heavy on some meshes (tetrahedra) $=>$ PolyMAC III keeps a **simpler top left block** but numerical cost **under control**.

- Same formulation as PolyMAC I
- Convection similar to PolyMAC I
- Gradients with HFV: add auxilary variables at the faces

Resulting linear system:

$$\begin{pmatrix} \frac{I}{\Delta t} + C(\mathbf{u}_f^t) & R & G_c^{HFV} & G_f^{HFV} \\ \tilde{R} & -\frac{1}{\nu} M_\omega & 0 & 0 \\ G_c^T & 0 & P_{cc} & P_{cf} \\ G_f^T & 0 & P_{fc} & P_{ff} \end{pmatrix} \begin{pmatrix} \mathbf{u}_f^{t+\Delta t} \\ \omega_e^{t+\Delta t} \\ \mathbf{p}_c^{t+\Delta t} \\ \mathbf{p}_f^{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \frac{I}{\Delta t} \mathbf{u}_f^t \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

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PolyMAC

Test problems

Numerical Results

Conclusions

2D and 3D test problems

In 2D: Bercovier-Engelman test for a right-hand-side of the form:

$$\vec{f} = \begin{pmatrix} f_1(x,y) + (y - \frac{1}{2}) \\ -f_1(y,x) + (x - \frac{1}{2}) \end{pmatrix}, \tag{6}$$

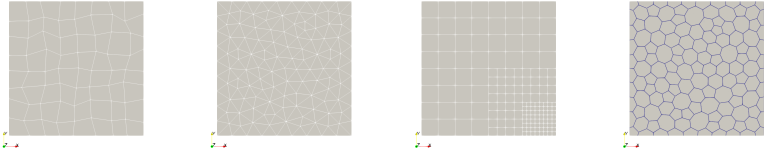
where $f_1(x,y) = 256[x^2(x-1)^2(12y-6) + y(y-1)(2y-1)(12x^2-12x+2)]$

In 3D: rotating Navier-Stokes problem ($\nu = 10^{-2}$, $\mathcal{R}e = 100$)

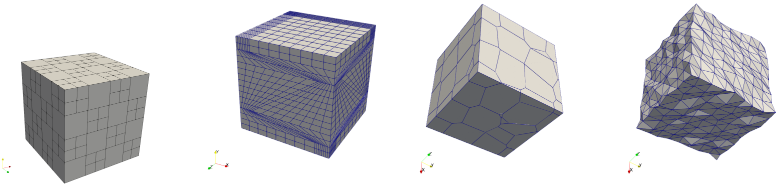
$$\begin{aligned} \vec{u} &= \begin{pmatrix} y - z \\ z - x \\ x - y \end{pmatrix} \\ p &= (x^2 + y^2 + z^2) - xy - xz - yz - \frac{1}{4} \end{aligned} \tag{7}$$

(Extensively used in FVCA benchmarks)

Meshes



(a) Quadrangles (b) Triangles (c) Locally Refined (d) Polygons



(e) Checkerboard (f) Kershaw (g) Voronoi (h) Random

Figure: Representation of some meshes from the benchmark.

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Convergence (in velocity)

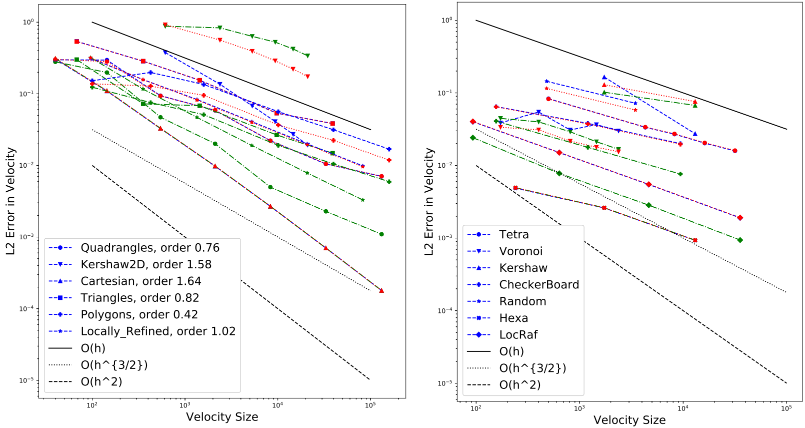

Figure: Left: 2D meshes; right: 3D meshes. PolyMAC I is represented in blue, PolyMAC II in green and PolyMAC III in red.

PolyMAC	Velocity I	II	III
Cartesian	1.64	**1.66**	**1.66**
Triangles	0.82	**1.32**	0.82
Quadrangles	0.76	**1.06**	0.78
Polygons	0.42	**0.74**	0.50
Locally Refined (2D)	1.02	**1.50**	1.04
Kershaw (2D)	**1.58**	0.44	0.86

Table: Estimated order of convergence for all three PolyMAC versions on the 2D meshes.

Convergence (cont'd)

PolyMAC	Velocity I	II	III
Hexa	1.08	**1.11**	**1.11**
Locally Refined (3D)	1.53	**1.71**	1.53
Kershaw (3D)	**2.67**	0.63	0.78
CheckerBoard	0.84	**1.23**	0.84
Voronoi	0.36	**1.02**	0.90
Tetrahedra	**1.23**	/	1.20
Random	**1.05**	/	**1.05**

Table: Estimated order of convergence for all three PolyMAC versions on the 3D meshes.

PolyMAC	Sparsity Ratio I	II	III
`Cartesian`	0.08	0.08	**0.03**
`Triangles`	0.05	0.11	**0.03**
`Quadrangles`	0.09	0.15	**0.06**
`Polygons`	0.06	0.07	**0.04**
`Locally Refined` (2D)	0.04	0.04	**0.02**
`Kershaw` (2D)	0.01	0.01	**0.00**

Table: Sparsity ratio on the 2D meshes.

PolyMAC	Sparsity Ratio I	II	III
Hexa	0.06	0.08	**0.04**
Locally Refined (3D)	0.08	0.07	**0.04**
Kershaw (3D)	0.01	0.01	**0.00**
CheckerBoard	0.08	0.06	**0.04**
Voronoi	0.16	0.13	**0.11**
Tetrahedra	0.02	/	**0.01**
Random	0.05	/	**0.03**

Table: Sparsity ratio on the 3D meshes.

Outline

Conclusions

PolyMAC	Accuracy	Computational cost	Stability
I	+	-	++
II	++	- -	(-)
III	+	++	(-)

(-) Sometimes divergent

Guidelines

- PolyMAC II most **accurate** BUT **heavy**
- PolyMAC III **as accurate** as PolyMAC I but **much sparser** ⇒ Default model
- PolyMAC I **always** convergent BUT **saddle point** ⇒ Back-up model

Perspectives

Future work:

- **Specific solver for the saddle point problem** (GMG or DD with AMG)
- Analysis of the loss of convergence: repair the method?
- **Fourth scheme in development**
- Add industrial cases to our benchmark

Thank you for your attention!