Stability estimates for Fortin-Soulie mixed finite elements

Improved stability estimates

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Outline

Outline

Context

- TrioCFD software:

  simulation of unsteady turbulent flows at low Mach number in industrial configurations.
- Solve in $(\mathbf{u}, p)$ (fluid velocity and pressure) such that:

$$\left\{ \begin{array}{rcl} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \mathbf{grad}\,)\mathbf{u} + \mathbf{grad}\, p & = & \mathbf{f} \\ \mathrm{div}\, \mathbf{u} & = & 0 \\ \mathbf{u}(t=0) & = & \mathbf{u}_0 \end{array} \right. , \text{ with appropriate BC.}$$

  Space discretization on a grid with MAC scheme or on simplexes with $\mathbf{P}^1_{nc} - (P^0 + P^1)$ FE.

  Resolution using a three-step scheme.
- R&D works on spatial numerical schemes.

  $\mathbf{P}^1_{nc} - P^0_{MPFA}$ scheme, with Andrew Peitavy (PhD in progress).

  DGFEM, with Mayssa Mroueh (PhD in progress).

  Improved stability estimates for solving Stokes problem with Fortin-Soulie finite elements
  Jamelot'23 (submitted).

Stokes Problem, variational formulation

- Solve in $(\mathbf{u}, p) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$ such that: $\left\{ \begin{array}{rcl} -\nu\,\Delta\mathbf{u} + \mathbf{grad}\,p & = & \mathbf{f}, \\ \mathrm{div}\,\mathbf{u} & = & 0. \end{array} \right.$

  $\mathbf{u} \in \mathbf{V}$ where: $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1_0(\Omega) \,|\, \mathrm{div}\,\mathbf{v} = 0\}, \quad \mathbf{H}^1_0(\Omega) = \mathbf{V} \oplus \mathbf{V}^\perp.$
- Let $\mathcal{X} = \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$, which is an Hilbert space endowed with the norm

$$\|(\mathbf{u}, p)\|_{\mathcal{X}} = \left(\|\mathbf{u}\|^2_{\mathbf{H}^1_0(\Omega)} + \nu^{-2}\,\|p\|^2_{L^2(\Omega)} \right)^{1/2}.$$

  $(FV - S)$ Solve in $(\mathbf{u}, p) \in \mathcal{X}$ such that $\forall (\mathbf{v}, q) \in \mathcal{X}$: $a_S\left((\mathbf{u}, p), (\mathbf{v}, q)\right) = \langle \mathbf{f},\, \mathbf{v} \rangle_{\mathbf{H}^1_0(\Omega)}$.

$$a_S : \left\{ \begin{array}{ccl} \mathcal{X} \times \mathcal{X} & \rightarrow & \mathbb{R} \\ (\,(\mathbf{u}', p'), (\mathbf{v}, q)\,) & \mapsto & \nu\,(\mathbf{u}', \mathbf{v})_{\mathbf{H}^1_0(\Omega)} - (\mathrm{div}\,\mathbf{v}, p')_{L^2(\Omega)} - (\mathrm{div}\,\mathbf{u}', q)_{L^2(\Omega)} \end{array} \right.$$

- Poincaré-Steklov inequality: $\exists C_{PS} > 0 \,|\quad \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \le C_{PS}\,\|\mathbf{Grad}\,v\|_{\mathbb{L}^2(\Omega)}$.

  $(\mathbf{v}, \mathbf{w})_{\mathbf{H}^1_0(\Omega)} = (\mathbf{Grad}\,v, \mathbf{Grad}\,w)_{\mathbb{L}^2(\Omega)}$ and $\|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)} = \|\mathbf{Grad}\,v\|_{\mathbb{L}^2(\Omega)}$.

- Girault-Raviart'86: the operator div is an isomorphism from $\mathbf{V}^\perp$ on $L^2_0(\Omega)$.

$$\forall q \in L^2_0(\Omega),\ \exists! \mathbf{v}_q \in \mathbf{V}^\perp \,|\quad \mathrm{div}\,\mathbf{v}_q = q, \quad \|\mathbf{v}_q\|_{\mathbf{H}^1_0(\Omega)} \le C_{\mathrm{div}}\,\|q\|_{L^2(\Omega)}.$$

Outline

Notations

- Let $\Omega \subset \mathbb{R}^d$ be a connected, bounded, open polygon or polyhedron with a Lipschitz boundary. Family of meshes $(\mathcal{T}_h)_h$, with $h \to 0$.
- Simplicial regular triangulation: $\mathcal{T}_h = \bigcup_K K$.

  Vertices set: $\mathcal{S}_h = \bigcup_S S$.

  Facets set: $\mathcal{F}_h = \bigcup_F F = \mathcal{F}_h^i \cup \mathcal{F}_h^b$; $\mathcal{F}_h^i$ (resp. $\mathcal{F}_h^b$): inner (resp. boundary) facets.

  Orthonormal vectors to facets: $(\mathbf{n}_F)_{F\in\mathcal{F}_h}$ ($\mathbf{n}_F$ outward oriented if $F \in \partial\Omega$).
- h_K: diameter of element K, ρ_K: diameter of the inscribed sphere of element K,

$$\exists \sigma > 0 \,|\, \forall h, \forall K \in \mathcal{T}_h, \quad \sigma_K = \frac{h_K}{\rho_K} \leq \sigma.$$

- For $d = 2$, $\sigma \geq \left(\frac{\pi}{2\,\theta}\right)^{1/2}$ where θ is the smallest angle of the triangulation.
- For $d = 3$, $\sigma \geq \left(\frac{\pi}{2\,\omega}\right)^{1/3}$ where ω is the smallest solid angle of the triangulation.

Bilinear form $(\mathbf{u}_h, \mathbf{v}_h)_h$ and operator div_h

- Piecewise regular functions.
- $\mathcal{P}_h H^1 = \{v \in L^2(\Omega) \,|\, \forall K \in \mathcal{T}_h,\ v_{|K} \in H^1(K)\}, \quad \mathcal{P}_h \mathbf{H}^1 = (\mathcal{P}_h H^1)^d.$

$$\forall v,\ w \in \mathcal{P}_h H^1, \quad (v, w)_h = \sum_{K \in \mathcal{T}_h} (\mathbf{grad}\, v, \mathbf{grad}\, w)_{\mathbf{L}^2(K)}, \quad \|v\|_h^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K)},$$

$$\forall \mathbf{v},\ \mathbf{w} \in \mathcal{P}_h \mathbf{H}^1, \quad (\mathbf{v}, \mathbf{w})_h = \sum_{K \in \mathcal{T}_h} (\mathbf{Grad}\, \mathbf{v}, \mathbf{Grad}\, \mathbf{w})_{\mathbb{L}^2(K)}, \quad \|\mathbf{v}\|_h^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{Grad}\, \mathbf{v}\|^2_{\mathbb{L}^2(K)}.$$

- Interface jumps for $v \in \mathcal{P}_h H^1$:

  For $F \in \mathcal{F}_h^i$, $\overline{F} = \overline{K}_L \cap \overline{K}_R$, $\mathbf{n}_F$ oriented from K_L to K_R $[v]_{|F} = \left(v_{|K_L} - v_{|K_R}\right)_{|F}$.

  For $F \in \mathcal{F}_h^b$, $[v]_{|F} = v_{|F}$.

- $\mathcal{P}_h \mathbf{H}(\mathrm{div}) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,|\, \forall K \in \mathcal{T}_h,\ \mathbf{v}_{|K} \in \mathbf{H}(\mathrm{div}\,; K)\}.$

  Operator div_h : $\forall \mathbf{v} \in \mathcal{P}_h \mathbf{H}(\mathrm{div})$, $\forall q \in L^2(\Omega)$, $\quad (\mathrm{div}_h \mathbf{v}, q) = \sum_{K \in \mathcal{T}_h} (\mathrm{div}\, \mathbf{v}, q)_{L^2(K)}.$

Nonconforming mixed finite elements (i) Crouzeix-Raviart'73

- Nonconforming approximation of $H^1(\Omega)$:

$$X_h = \left\{ v \in L^2(\Omega) \,|\, \forall K \in \mathcal{T}_h \quad v_{|K} \in P^k(K) \quad \text{and} \quad \forall F \in \mathcal{F}_h^i, \forall q \in P^{k-1}(F) \quad \int_F [v]\, q = 0 \right\}.$$

- Nonconforming approximation of $H^1_0(\Omega)$:

$$X_{0,h} = \left\{ v \in X_h \,|\, \forall F \in \mathcal{F}_h^b, \forall q \in P^{k-1}(F) \quad \int_F v\, q = 0 \right\}.$$

- Thanks to the patch test, on can prove that:
  The mapping $v_h \to \|v_h\|_h$ is a norm on $X_{0,h}$.
  We have a discrete Poincaré-Steklov inequality Ern-Guermond'21, Lemma 36.6:

$$\forall v \in X_{0,h}, \quad \|v\|_{L^2(\Omega)} \leq C_{PS}^{nc} \|v\|_h.$$

- Approximation of $\mathcal{X} = \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$:
  Nonconforming approximation of $\mathbf{H}^1(\Omega)$ and $\mathbf{H}^1_0(\Omega)$: $\mathbf{X}_h = (X_h)^d$ and $\mathbf{X}_{0,h} = (X_{0,h})^d$.
  Approximation of $L^2_0(\Omega)$: $\quad Q_h = \{ q \in L^2_0(\Omega) \,|\, \forall K \in \mathcal{T}_h, \quad q_{|K} \in P^{k-1}(K) \}$.

$$\mathcal{X}_h := \mathbf{X}_{0,h} \times Q_h, \quad \|(\mathbf{v}, q)\|_{\mathcal{X}_h} = \left(\|\mathbf{v}\|_h^2 + \nu^{-2} \|q\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Nonconforming mixed finite elements (*ii*) Crouzeix-Raviart'73

- $(FV_h - S)$ Find $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$ such that $\forall(\mathbf{v}_h, q_h) \in \mathcal{X}_h$: $a_{S,h}\left((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\right) = \ell_{\mathbf{f}}(\mathbf{v}_h)$.

$$a_{S,h} : \left\{ \begin{array}{rcl} \mathcal{X}_h \times \mathcal{X}_h & \to & \mathbb{R} \\ \left((\mathbf{u}'_h, p'_h), (\mathbf{v}_h, q_h)\right) & \mapsto & \nu\,(\mathbf{u}'_h, \mathbf{v}_h)_h - (\mathrm{div}_h\, \mathbf{v}_h, p'_h) - (\mathrm{div}_h\, \mathbf{u}'_h, q_h) \end{array} \right.$$

For $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\ell_{\mathbf{f}}(\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)}$.

For $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $\ell_{\mathbf{f}}(\mathbf{v}_h) = \langle\, \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h)\,\rangle_{\mathbf{H}^1_0(\Omega)}$ Veeser-Zanotti'18.

- Suppose there exists a Fortin operator $\Pi_{nc} : \mathbf{H}^1(\Omega) \to \mathbf{X}_h$ such that

$$\left\{ \begin{array}{rrcl} \forall \mathbf{v} \in \mathbf{H}^1(\Omega) & (\mathrm{div}_h\, \Pi_{nc}\mathbf{v}, q) & = & (\mathrm{div}\, \mathbf{v}, q)_{L^2(\Omega)}, \quad \forall q \in Q_h, \\ \exists \widetilde{C} > 0 \,|\, \forall \mathbf{v} \in \mathbf{H}^1(\Omega) & \|\Pi_{nc}\mathbf{v}\|_h & \leq & \widetilde{C}\, \|\mathbf{Grad}\, \mathbf{v}\|_{\mathbb{L}^2(\Omega)}. \end{array} \right.$$

Let $p'_h \in Q_h \subset L^2_0(\Omega)$. Consider $\mathbf{v}_{p'_h} \in \mathbf{V}^\perp$ such that

$$\mathrm{div}\, \mathbf{v}_{p'_h} = p'_h \quad \text{and} \quad \|\mathbf{v}_{p'_h}\|_{\mathbf{H}^1_0(\Omega)} \leq C_{\mathrm{div}}\, \|p'_h\|_{L^2(\Omega)}.$$

Let $\mathbf{v}_{h,p'_h} = \Pi_{nc}\mathbf{v}_{p'_h}$. We have, setting $\widetilde{C}_{\mathrm{div}} = \widetilde{C}\, C_{\mathrm{div}}$:

$$\forall q \in Q_h\, (\mathrm{div}_h\, \mathbf{v}_{h,p'_h}, q) = (p'_h, q)_{L^2(\Omega)} \quad \text{and} \quad \|\mathbf{v}_{h,p'_h}\|_h \leq \widetilde{C}_{\mathrm{div}}\, \|p'_h\|_{L^2(\Omega)}.$$

Mixed finite elements $\mathbf{P}^1_{nc} - P^0$ Crouzeix-Raviart'73, Example 4

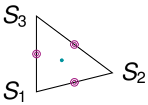
- Approximation of $\mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$: $\mathcal{X}_{cr} = \mathbf{X}_{cr} \times Q_{cr}$.

$$\mathbf{X}_{cr} = (X_{cr})^d, \quad X_{cr} = \left\{ v_h \in L^2(\Omega) \,|\, \forall K \in \mathcal{T}_h \quad v_{h|K} \in P^1(K);\ \forall F \in \mathcal{F}_h \quad \int_F [v_h] = 0 \right\},$$

$$Q_{cr} = \left\{ q_h \in L^2_0(\Omega) \,|\, \forall K \in \mathcal{T}_h \quad q_{h|K} \in P^0(K) \right\}.$$

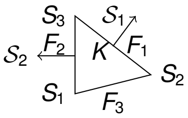
$$X_{cr} = \mathrm{vect}\left((\psi_F)_{F \in \mathcal{F}_h} \right), \text{ with:} \quad \psi_{F|K} := \begin{cases} 0 & \text{si } F \notin \partial K, \\ 1 - d\lambda_{S_F,K} & \text{si } F \in \partial K. \end{cases}$$

- Degrees of freedom:



Velocity, pressure.

Barycentric coordinates:



$$\lambda_{S_i,K} = (d|K|)^{-1}(\vec{OS_i} - \mathbf{x}) \cdot \mathcal{S}_{F_i,K}.$$

We have: $\mathbf{grad}\, \psi_{F|K} = -d\, \mathbf{grad}\, \lambda_{S_F,K} = |K|^{-1} \mathcal{S}_{F,K}$.

Fortin operator $\mathbf{P}^1_{nc} - P^0$

- Interpolation operator for scalar fonctions: $\pi_{cr} : H^1(\Omega) \to X_{cr}$, $\pi_{cr} v = \sum_{F\in\mathcal{F}_h} \left(\frac{1}{|F|} \int_F v \right) \psi_F$.

  Apel-Nicaise-Schöberl'01, Lemma 2:

$$\mathbf{grad}\, \pi_{cr} v_{|K} = |K|^{-1} \int_K \mathbf{grad}\, \pi_{cr} v = |K|^{-1} \int_{\partial K} \pi_{cr} v\, \mathbf{n} = |K|^{-1} \int_{\partial K} v\, \mathbf{n} = |K|^{-1} \int_K \mathbf{grad}\, v.$$

  Hence: $|\mathbf{grad}\, \pi_{cr} v_{|K}| \leq |K|^{-1/2} \|\mathbf{grad}\, v\|_{L^2(K)} \Rightarrow \quad \|\mathbf{grad}\, \pi_{cr} v\|_{L^2(K)} \leq \|\mathbf{grad}\, v\|_{L^2(K)}$.

  For all $v \in H^1(\Omega)$, we have: $\|\pi_{cr} v\|_h \leq \|\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}$.

- Fortin operator / Interpolation operator for vector fonctions: $\Pi_{cr}\mathbf{v} = (\, \pi_{cr} v_i \,)_{i=1}^d$.

$$\forall K \in \mathcal{T}_h \int_K \operatorname{div} \Pi_{cr}\mathbf{v} = \int_K \sum_{F\in\partial K} |K|^{-1} \frac{\int_F \mathbf{v}\cdot \mathcal{S}_{F,K}}{|F|} = \sum_{F\in\partial K} \int_F \mathbf{v}\cdot \mathbf{n}_{F,K} = \int_K \operatorname{div} \mathbf{v}.$$

  For all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $q_h \in Q_{cr}$, we have:

$$\|\Pi_{cr}\mathbf{v}\|_h \leq \|\mathbf{Grad}\, \mathbf{v}\|_{\mathbb{L}^2(\Omega)} \quad \text{and} \quad (\operatorname{div}_h \Pi_{cr}\mathbf{v}, q_h) = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}$$

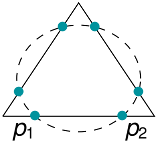
Mixed finite elements $\mathbf{P}^2_{nc} - P^1_{disc}$ Fortin-Soulie'83

- Approximation of $\mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$: $\mathcal{X}_{fs} = \mathbf{X}_{fs} \times Q_{fs}$.

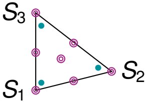
$$\mathbf{X}_{fs} = (X_{fs})^d,\ X_{fs} = X_{0,lg} \oplus \Phi_h, \qquad Q_{fs} = \left\{ q_h \in L^2_{zvm}(\Omega) \,|\, \forall K \in \mathcal{T}_h,\quad q_h \in P^1(K) \right\}.$$

We set:
$$\begin{cases} X_{0,lg} = \left\{ v_h \in H^2(\Omega) \,|\, \forall K \in \mathcal{T}_h,\quad v_{h|K} \in P^2(K),\ v_{h|\partial\Omega=0} \right\}, \\ \Phi_h = \left\{ \phi_h \in L^2(\Omega) \,|\, \phi_{h|K} = \alpha_K\, \phi_K,\ \alpha_K \in \mathbb{R} \right\} \text{ where } \phi_K = 2 - 3 \sum_{S \in \partial K} \lambda^2_{S,K}. \end{cases}$$

- Gauss-Legendre points:



Degrees of freedom:



Velocity, pressure.

- Integration: $\forall q \in P^3(F),\ \int_F q = \frac{|F|}{2}\,(\,q(p_1) + q(p_2)\,)$.

$$\forall F \in \partial K,\ \forall q \in P^1(F),\ \int_F \phi_K\, q = 0.$$

Fortin operator $\mathbf{P}^2_{nc} - P^1_{disc}$

- Interpolation operator for scalar functions:

$$\tilde{\pi}_{fs} : H^1(\Omega) \to X_{fs} \mid \left\{ \begin{array}{llcl} \forall S \in \mathcal{V}_h, & (\tilde{\pi}_{fs}(v))(S) & = & (\pi_{Scott-Zhang}(v))(S) \\ \forall F \in \mathcal{F}_h, & \int_F \tilde{\pi}_{fs}(v) & = & \int_F v \\ \forall K \in \mathcal{T}_h, & \int_K \tilde{\pi}_{fs}(v) & = & \int_K v \end{array} \right. .$$

- Interpolation operator for vector functions: $\widetilde{\Pi}_{fs} : \mathbf{H}^1(\Omega) \to \mathbf{X}_{fs} \mid \widetilde{\Pi}_{fs}(\mathbf{v}) = {}^t(\tilde{\pi}_{fs}(v_x), \tilde{\pi}_{fs}(v_y))$.
- Fortin operator:

$$\Pi_{fs} : \mathbf{H}^1(\Omega) \to \mathbf{X}_{fs} \mid \left\{ \begin{array}{llcl} \forall S \in \mathcal{V}_h, & (\Pi_{fs}(\mathbf{v}))(S) & = & (\Pi_{Scott-Zhang}(\mathbf{v}))(S) \\ \forall F \in \mathcal{F}_h, & \int_F \Pi_{fs}(\mathbf{v}) & = & \int_F \mathbf{v} \\ \forall K \in \mathcal{T}_h, & \int_K \mathbf{x}\, \mathrm{div}\, \Pi_{fs}(\mathbf{v}) & = & \int_K \mathbf{x}\, \mathrm{div}\, \mathbf{v} \end{array} \right. .$$

- For all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $q_h \in Q_{fs}$, we have: $(\mathrm{div}_h \Pi_{fs}\mathbf{v}, q_h) = (\mathrm{div}\, \mathbf{v}, q_h)_{L^2(\Omega)}$.
- Theorem: Let $\sigma_D > 0$. For all $\mathbf{v} \in \bigcap_{0<s<\sigma_D} \mathbf{H}^{1+s}(\Omega)$, we have:

$$\begin{array}{llrcl} \forall s \in]0, \sigma_D[, & \forall K \in \mathcal{T}_h, & \|\mathbf{Grad}\,(\Pi_{fs}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K)} & \leq & (\sigma_K)^2\,(h_K)^s\,|\mathbf{v}|_{1+s,K}, \\ \forall s \in]0, \sigma_D[, & \exists C_{FS} = \mathcal{O}(\sigma^2), & \|\Pi_{fs}\mathbf{v} - \mathbf{v}\|_h & \leq & C_{FS}\, h^s\, |\mathbf{v}|_{1+s,\Omega}. \end{array}$$

Stability constant of Fortin operator $\mathbf{P}^2_{nc} - P^1_{disc}$ (i)

- $\|\mathbf{Grad}\,(\Pi_{fs}(\mathbf{v}) - \mathbf{v})\|_{\mathbb{L}^2(K)} \lesssim \|\mathbf{Grad}\,\left(\Pi_{fs}(\mathbf{v}) - \widetilde{\Pi}_{fs}(\mathbf{v})\right)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{Grad}\,\left(\widetilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}\right)\|_{\mathbb{L}^2(K)}.$
- For all $\hat{\mathbf{v}} \in \mathbf{P}^2(\hat{K})$, $\widehat{\widetilde{\Pi}}_{fs}(\hat{\mathbf{v}}) = \hat{\mathbf{v}}$.

  Thanks to Bramble-Hilbert/Deny-Lions Lemma, we have Ern-Guermond'21, Lemma 11.9:

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad \|\mathbf{Grad}\,\left(\widetilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}\right)\|_{\mathbb{L}^2(K)} \quad \lesssim \quad \sigma_K\,|\mathbf{v}|_{1,K},$$

$$\forall \mathbf{v} \in \mathbf{H}^2(\Omega), \quad \|\mathbf{Grad}\,\left(\widetilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}\right)\|_{\mathbb{L}^2(K)} \quad \lesssim \quad \sigma_K\,h_K\,|\mathbf{v}|_{2,K}.$$

- Using interpolation theory Tartar'07, Lemma 22.2:

$$\forall \mathbf{v} \in \bigcap_{0<s<\sigma_D} \mathbf{H}^{1+s}(\Omega), \quad \|\mathbf{Grad}\,\left(\widetilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}\right)\|_{\mathbb{L}^2(K)} \lesssim \sigma_K\,(h_K)^s\,|\mathbf{v}|_{1+s,K}.$$

- $\widetilde{\Pi}_{fs}(\mathbf{v}) = \sum_{S\in\mathcal{V}_h} \mathbf{v}_S\phi_S + \sum_{F\in\mathcal{F}_h} \mathbf{v}_F\phi_F + \sum_{K\in\mathcal{T}_h} \tilde{\mathbf{v}}_K\phi_K, \quad \Pi_{fs}(\mathbf{v}) = \sum_{S\in\mathcal{V}_h} \mathbf{v}_S\phi_S + \sum_{F\in\mathcal{F}_h} \mathbf{v}_F\phi_F + \sum_{K\in\mathcal{T}_h} \mathbf{v}_K\phi_K.$

Stability constant of Fortin operator $\mathbf{P}^2_{nc} - P^1_{disc}$ (ii)

- We have: $\Pi_{fs}(\mathbf{v}) - \widetilde{\Pi}_{fs}(\mathbf{v}) = (\mathbf{v}_K - \tilde{\mathbf{v}}_K)\,\phi_K$, hence:

$$\begin{aligned} \|\mathbf{Grad}\left(\Pi_{fs}(\mathbf{v}) - \widetilde{\Pi}_{fs}(\mathbf{v})\right)\|_{\mathbb{L}^2(K)} &= \|\mathbf{Grad}\,(\,(\mathbf{v}_K - \tilde{\mathbf{v}}_K)\,\phi_K\,)\|_{\mathbb{L}^2(K)} \\ &\lesssim \|\mathbf{grad}\,\phi_K\|_{\mathbf{L}^2(K)}\,|\mathbf{v}_K - \tilde{\mathbf{v}}_K|, \\ &\lesssim \sigma_K\,|\mathbf{v}_K - \tilde{\mathbf{v}}_K|. \end{aligned}$$

- Let us estimate $|\mathbf{v}_K - \tilde{\mathbf{v}}_K|$.

$$\int_K \left(\Pi_{fs}(\mathbf{v}) - \widetilde{\Pi}_{fs}(\mathbf{v})\right) = \int_K \phi_K\,(\mathbf{v}_K - \tilde{\mathbf{v}}_K) = \frac{|K|}{4}\,(\mathbf{v}_K - \tilde{\mathbf{v}}_K)$$

$$\int_K \left(\Pi_{fs}(\mathbf{v}) - \widetilde{\Pi}_{fs}(\mathbf{v})\right) = \int_K (\Pi_{fs}(\mathbf{v}) - \mathbf{v}) = \int_{\partial K} \mathbf{x}\,(\Pi_{fs}(\mathbf{v}) - \mathbf{v}) \cdot \mathbf{n}_{|\partial K} = \int_{\partial K} \mathbf{x}\,\left(\widetilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}\right) \cdot \mathbf{n}_{|\partial K}$$

- Lemma $(\star)$: Let $\mathbf{w} \in \mathbf{H}^1(\Omega)$ and $q \in P^1(K)$. Setting $\underline{\mathbf{w}} = \mathbf{w}_{|K} - \frac{\int_K \mathbf{w}}{|K|}$, we have:

$$\left|\int_{\partial K} q\,\underline{\mathbf{w}} \cdot \mathbf{n}\right| \leq |K|\,\|\mathbf{Grad}\,\underline{\mathbf{w}}\|_{\mathbb{L}^2(K)}.$$

Stability constant of Fortin operator $\mathbf{P}^2_{nc} - P^1_{disc}$ (iii)

- Using Lemma $(\star)$, we get:

$$|\mathbf{v}_K - \tilde{\mathbf{v}}_K| = 4\,|K|^{-1}\left|\int_{\partial K}\mathbf{x}\left(\widetilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}\right)\cdot\mathbf{n}_{|\partial K}\right| \leq 4\,\|\mathbf{Grad}\left(\widetilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}\right)\|_{\mathbb{L}^2(K)}.$$

- We deduce that $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$:

$$\|\mathbf{Grad}\left(\Pi_{fs}(\mathbf{v}) - \widetilde{\Pi}_{fs}(\mathbf{v})\right)\|_{\mathbb{L}^2(K)} \lesssim \sigma_K\,|\mathbf{v}_K - \tilde{\mathbf{v}}_K| \lesssim \sigma_K\,\|\mathbf{Grad}\left(\widetilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}\right)\|_{\mathbb{L}^2(K)}.$$

- Stability constant for Π_{fs}: $\forall \mathbf{v} \in \bigcap_{0<s<\sigma_D} \mathbf{H}^{1+s}(\Omega)$,

$$\begin{aligned}
\|\mathbf{Grad}\,(\Pi_{fs}(\mathbf{v}) - \mathbf{v})\|_{\mathbb{L}^2(K)} &\lesssim (\sigma_K + 1)\,\|\mathbf{Grad}\left(\widetilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}\right)\|_{\mathbb{L}^2(K)},\\
&\lesssim (\sigma_K)^2\,(h_K)^s\,|\mathbf{v}|_{1+s,K}.\\
\|\mathbf{Grad}\,(\Pi_{fs}(\mathbf{v}) - \mathbf{v})\|_{\mathbb{L}^2(\Omega)} &\lesssim \sigma^2\,h^s\,|\mathbf{v}|_{1+s,\Omega} \text{ by summation.}
\end{aligned}$$

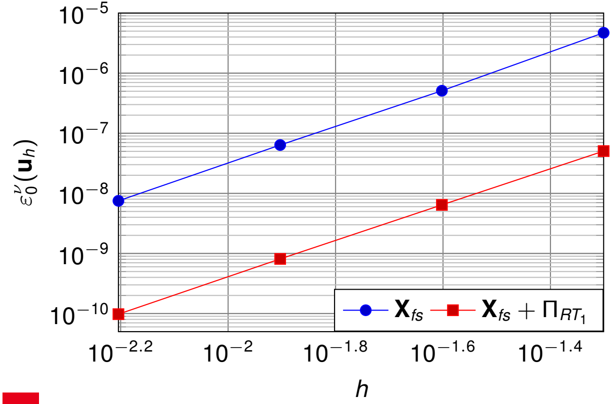
Outline

Numerical results with a regular solution

- Stokes Problem in $\Omega = (0,1)^2$ with:

$$\mathbf{u} = \begin{pmatrix} (1-\cos(2\pi x))\sin(2\pi y) \\ (\cos(2\pi y)-1)\sin(2\pi x) \end{pmatrix}, \quad p = \sin(2\pi x)\sin(2\pi y), \quad \mathbf{f} = -\nu\,\Delta\mathbf{u} + \mathbf{grad}\,p.$$

- Plots $\varepsilon_0^\nu(\mathbf{u}) = \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}/\|(\mathbf{u},p)\|_{\mathcal{X}}$ for $\nu = 1.00\,e-4$. Convergence rate: $\tau = 3$.



$\mathbf{X}_{fs}$: $\mathbf{u}_h \notin \mathbf{H}(\mathrm{div}\,;\Omega)$.

$\mathbf{X}_{fs} + \Pi_{RT_1}$: $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}\,;\Omega)$.

Pressure robust disc.:
Linke et al

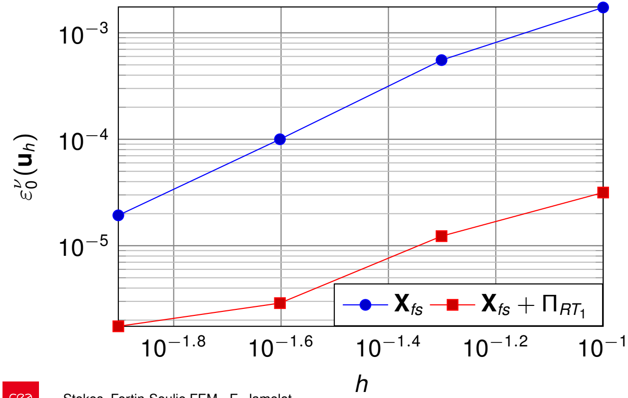
Π_{RT_1} computation:
Gatica'14, Lemma 3.11

Numerical results with a low-regular solution

- Stokes Problem in $\Omega = (0,1)^2$. We set: $\mathbf{x}_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$, $(r, \theta) = \left(|\mathbf{x} - \mathbf{x}_0|, \arctan\left(\frac{y-y_0}{x-x_0}\right) \right)$.

  Consider the prescribed solution (with inhomogeneous Dirichlet BC):

$$\mathbf{u} = r^\alpha \, \mathbf{e}_\theta, \quad p = r^\beta - \int_\Omega r^\beta, \quad \mathbf{f} = -\nu(\alpha^2 - 1)\rho^{\alpha-2}\mathbf{e}_\theta + \beta\, \mathbf{e}_r.$$



$\alpha = 0.45$ and $\beta = 1$,

$\nu = 1.00\,e - 4$,

$$\varepsilon_0^\nu(\mathbf{u}) = \frac{\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}}{\|(\mathbf{u}, p)\|_{\mathcal{X}}}.$$

$\tau_{fs} = 2.18$,

$\tau_{fs+rt} = 1.39$.

Outline

Context

Nonconforming discretization

Numerical results

Conclusion

Conclusion

- Local stability estimates:

  For order 1 in $2D$ and $3D$ without mesh regularity assumption.

  For order 2 in $2D$ with mesh regularity assumption.

- Higher order and higher dimension:

  Heavy mathematical material needed Sauter et al.

  Splitting the normal and tangential components could help to get (local) stability estimates.

- Numerical results could be enhanced using BDM projection instead of RT.

  Brezzi-Douglas-Marini'85, Boffi-Brezzi-Fortin'13.