Multiplicity Result for a Class of Nonlinear Elliptic System in Variable Exponent Sobolev Spaces

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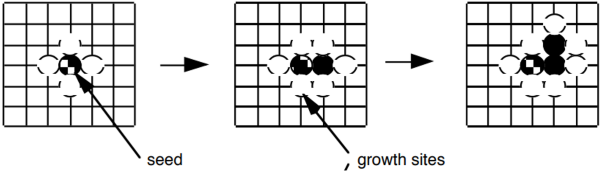
Présentation du problème: systèmes de Diffusion (D)



$$\frac{\partial [X_i]}{\partial t} = \mu_i \left([X_i]_R - 2[X_i] + [X_i]_L \right)$$

$$\Downarrow$$

$$\frac{\partial [X_i]}{\partial t} = \Delta [X_i]$$

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$$\frac{\partial [X_i]}{\partial t} = \Delta_{p_i(x)}[X_i] = div\left(|\nabla [X_i]|^{p_i(x)-2} \nabla [X_i] \right)$$

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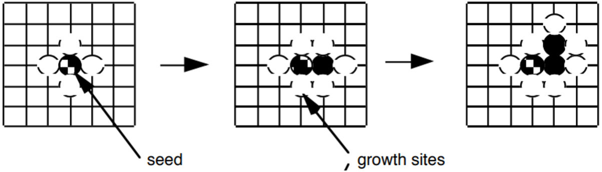

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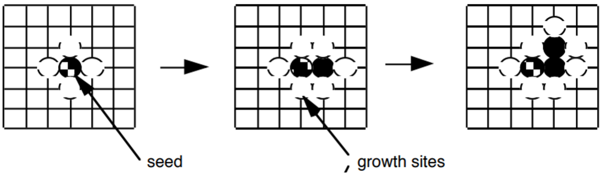

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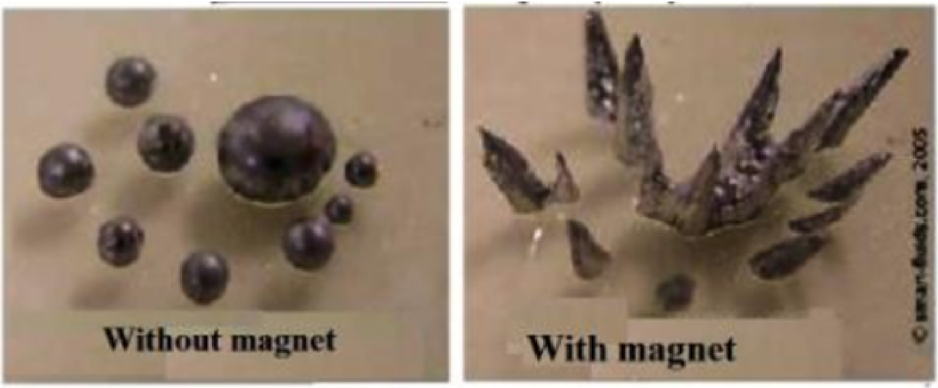
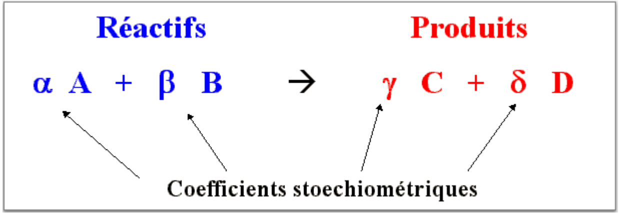


Présentation du problème: systèmes de Réaction-Diffusion (RD)

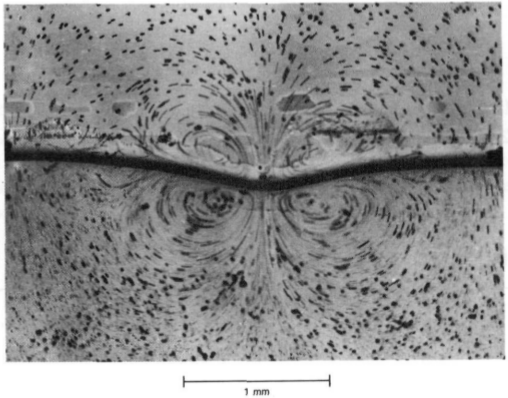
Solution de type **GEL**: $\boxed{\alpha_i X_1 + \beta_i X_2 \to \text{produits}_i}$



$$\boxed{\frac{\partial[X_i]}{\partial t} = \Delta_{p_i(x)}[X_i] + c_i[X_1]^{\alpha_1}[X_2]^{\beta_i}}$$

Présentation du problème: systèmes de Réaction-Diffusion-Convection (RDC)

Solution **PAS** de type **GEL**:



$$\frac{\partial[X_i]}{\partial t} = \Delta_{p_i(x)}[X_i] + k_i[X_1]^{\alpha_i}[X_2]^{\beta_i} + d_{i,1}\,|\nabla[X_1]|^{\gamma_i} + d_{i,2}\,|\nabla[X_2]|^{\overline{\gamma}_i}$$

Présentation du problème

Systèmes de Réaction-Diffusion-Convection:

$$(S)\begin{cases} -\Delta_{p_i(x)} u_i = f_i(x, u_1, .., u_n, \nabla u_1, .., \nabla u_n) & \text{dans } \Omega \\ u_i = 0 & \text{sur } \partial\Omega \\ 1 \leq i \leq n \end{cases}$$

n=1: scalaire

n=2: vectoriel

Termes sources:

$$|f_i(x, s_1, .., s_n, \xi_1, .., \xi_n)| \leq c_i(x) s_1^{\alpha_i(x)} s_2^{\beta_i(x)} + d_i(x) |\xi_1|^{\gamma_i(x)} + e_i(x) |\xi_2|^{\overline{\gamma}_i(x)},$$

$$(x, s_i, \xi_i) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$$

Présentation du problème

Systèmes de Réaction-Diffusion-Convection:

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$$(x, s_i, \xi_i) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$$

Espace fonctionnel de recherche

Espaces d'Orlicz-Sobolev:

$$\boxed{\prod_{i=1}^{n} W_0^{1,p_i(x)}(\Omega)}$$

[1] X. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W_m^{p(x)}(\Omega)$*, Journal of mathematical analysis and applications, **263.2** (2001), 424-446.

Banks' Mean Value Theorem

Theorem

Let f and ϕ be real valued functions defined for $x \in \Omega$ with f integrable over Ω and

$$-\infty < m \leq \phi(x) \leq M < \infty.$$

Let $\Omega(y) = \{x \in \Omega \,|\, \phi(x) \geq y\}$. If

$$0 \leq \int_{\Omega(y)} f(x)dx \leq \int_{\Omega} f(x)dx$$

for all $y \in [m, M]$, then there exists a number $\gamma \in [m, M]$ such that

$$\gamma \int_{\Omega} f(x)dx = \int_{\Omega} f(x)\phi(x)dx.$$

Proof of Banks' Theorem I

Step 1: Banks proves the following equalities

- $$\int_\Omega f(x)\phi(x)dx = m\int_\Omega f(x)dx + \int_m^M \left(\int_{\Omega(y)} f(x)dx\right) dy$$
- $$\int_\Omega f(x)\phi(x)dx = M\int_\Omega f(x)dx - \int_m^M \left(\int_{\Omega(y)^C} f(x)dx\right) dy$$

with

$$\Omega(y) = \{x \in \Omega \,|\, \phi(x) \geq y\}.$$

Step 2: He assumes that

Proof of Banks' Theorem II

- $\int_m^M \left(\int_{\Omega(y)} f(x)dx \right) dy \geq 0$
- $\int_m^M \left(\int_{\Omega(y)^C} f(x)dx \right) dy = (M-m) \int_\Omega f(x)dx - \int_m^M \left(\int_{\Omega(y)} f(x)dx \right) dy \geq 0$

Step 3: He deduces that

$$m \int_\Omega f(x)dx \leq \int_\Omega f(x)\phi(x)dx \leq M \int_\Omega f(x)dx$$

[1] D. Banks, *An integral inequality*, Proceedings of the American Mathematical Society 5 (14) (1963), 823-828.

Adaptated Mean Value Theorem

Theorem

Let $u \in W_0^{1,p(x)}(\Omega)$ be the solution of a nonlinear elliptic equation of the form

$$-\Delta_{p(x)} u = h(x) \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega, \tag{1}$$

where h is a sign-constant function. Let $f : \Omega \to \mathbb{R}$ be a Lipschitz continuous function satisfying $-\infty < m \leq \phi(x) \leq M < \infty$ for some constants m, M. Then, for any sign-constant function $\phi \in W_0^{1,p(x)}(\Omega)$, there exists a real $\gamma \in [m, M]$, depending on φ, such that

$$\int_\Omega \phi(x)|\nabla u|^{p(x)-2}\nabla u \nabla\varphi dx = \gamma \int_\Omega h(x)\phi dx. \tag{2}$$

Proof of the Adaptated Mean Value Theorem I

Step 1: We prove the following equalities

- $$\int_\Omega a_{(\epsilon,\Omega)}(x)\phi(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\varphi_\epsilon dx = m\int_\Omega a_{(\epsilon,\Omega)}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\varphi_\epsilon dx + \int_m^M\left(\int_\Omega a_{(\epsilon,\Omega(y))}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\varphi_\epsilon dx\right)dy$$
- $$\int_\Omega a_{(\epsilon,\Omega)}(x)\phi(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\varphi_\epsilon dx = M\int_\Omega a_{(\epsilon,\Omega)}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\varphi_\epsilon dx - \int_m^M\left(\int_\Omega a_{(\epsilon,\Omega(y)^c)}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\varphi_\epsilon dx\right)dy$$

with

$$a_{(\epsilon,\Omega(y))}(x) = \mathbb{1}_{\Omega(y)\cap\Omega_\epsilon}\star\Psi_\epsilon$$

Proof of the Adaptated Mean Value Theorem II

Step 2: We prove that

- $$\int_m^M \left(\int_\Omega a_{(\epsilon,\Omega(y))}(x) \, |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_\epsilon dx \right) dy \geq 0$$
- $$\int_m^M \left(\int_\Omega a_{(\epsilon,\Omega(y)^C)}(x) \, |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_\epsilon dx \right) dy \geq 0$$

To do so, we prove the existence of $v_{(\phi_\epsilon,y)}$ such that

$$\nabla v_{(\phi_\epsilon,y)} = a_{(\epsilon,A(y))} \nabla \varphi_\epsilon \quad \text{a.e. in } \Omega. \tag{3}$$

Proof of the Adaptated Mean Value Theorem III

which implies that

$$\int_m^M \left(\int_\Omega a_{(\epsilon, A(y))}(x) \, |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_\epsilon dx \right) dy = \int_m^M \left(\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v_{(\phi_\epsilon, y)} dx \right) dy$$
$$= \int_m^M \left(\int_\Omega h(x) v_{(\phi_\epsilon, y)}(x) dx \right) dy \geq 0.$$

Step 3: We deduce that

$$m \int_\Omega a_{(\epsilon,\Omega)}(x) \, |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_\epsilon dx \leq \int_\Omega a_{(\epsilon,\Omega)}(x) \phi(x) \, |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_\epsilon dx$$
$$\leq M \int_\Omega a_{(\epsilon,\Omega)}(x) \, |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_\epsilon dx$$

Proof of the Adaptated Mean Value Theorem IV

that is, doing $\epsilon \to 0$,

$$m\int_\Omega h(x)\varphi dx = m\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx \le \int_\Omega \phi(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx$$
$$\le M\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx = M\int_\Omega h(x)\varphi dx$$

[1] K. Perera & E.A. Silva, *Existence and multiplicity of positive solutions for singular quasilinear problems*, Journal of mathematical analysis and applications **323** (2006), 1238–1252.

[2] D.D. Hai, *Singular boundary value problems for the p-Laplacian*, Nonlinear Analysis: Theory, Methods & Applications **73** (2010), 2876–2881.

Application 1

Lemma

Let $h \in L^\infty(\Omega)$ be a function such that $\|h\|_\infty \le 1$, and let $u \in W_0^{1,p(x)}(\Omega)$ be the weak solution of the Dirichlet problem

$$-\Delta_{p(x)} u = h(x) \text{ in } \Omega, \;\; u = 0 \text{ on } \partial\Omega. \tag{4}$$

Then, there exists a constant $\bar{k}_p > 0$ and $\tau \in (0,1)$, depending only on p, N, and Ω, such that

$$\|u\|_{1,\tau} \le \bar{k}_p \|h\|_\infty^{\frac{1}{p^\pm - 1}} \tag{5}$$

with

$$p^\pm := \begin{cases} p^- & \text{if } \|h\|_\infty > 1 \\ p^+ & \text{if } \|h\|_\infty \le 1. \end{cases}$$

Proof of Application 1 I

By the MVT, there exists $x_0 \in \Omega$ such that, for $\varphi \in \left(W_0^{1,p(.)}(\Omega)\right)_+$,

$$\begin{aligned}
&\int_\Omega |\nabla(\|h\|_\infty^{\frac{-1}{p^\pm-1}} u)|^{p(x)-2}\nabla(\|h\|_\infty^{\frac{-1}{p^\pm-1}} u)\nabla\varphi\, dx \\
&= \int_\Omega \|h\|_\infty^{\frac{-(p(x)-1)}{p^\pm-1}} |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\, dx \\
&= \|h\|_\infty^{\frac{-(p(x_0)-1)}{p^\pm-1}} \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\, dx = \|h\|_\infty^{\frac{-(p(x_0)-1)}{p^\pm-1}} \int_\Omega h(x)\varphi\, dx \\
&\leq \|h\|_\infty^{\frac{-(p^\pm-1)}{p^\pm-1}} \int_\Omega h(x)\varphi\, dx = \int_\Omega \|h\|_\infty^{-1} h(x)\varphi\, dx \leq \int_\Omega \varphi\, dx.
\end{aligned}$$

Thus,

$$-\Delta_{p(x)}(\|h\|_\infty^{\frac{-1}{p^\pm-1}} u) \leq 1.$$

Proof of Application 1 II

By Fan's regularity theorem, there exists $\tau \in (0,1)$ and $\bar{k}_p > 0$ such that

$$\|h\|_{\infty}^{\frac{-1}{p^{\pm}-1}} \|u\|_{C^{1,\tau}(\overline{\Omega})} \leq \bar{k}_p.$$

[1] X. Fan, *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, Journal of Inequalities and Applications 235 (2) (2007), 397-417.

Application 2

Let $\zeta_i \in C^{1,\tau}(\overline{\Omega})$, $\tau \in (0,1)$, be the solutions of the Dirichlet problems

$$-\Delta_{p_i(x)}\zeta_i = m_i d(x)^{\alpha_i(x)+\beta_i(x)} \text{ in } \Omega, \quad \zeta_i(x) = 0 \text{ on } \partial\Omega. \tag{6}$$

Proposition

Assume that $0 \leq \min\{\alpha_i^-, \beta_i^-\} \leq \alpha_i^+ + \beta_i^+ \leq p_i^- - 1$. *Then it holds*

$$-\Delta_{p_i(x)}\underline{u}_i \leq m_i\, \underline{u}_1^{\alpha_i(x)}\, \underline{u}_2^{\beta_i(x)} \text{ in } \Omega, \tag{7}$$

where $\underline{u}_i = c_i\zeta_i$, *provided that* $c_i > 0$ *is sufficiently small.*

Proof of Application 2 I

From the assumption $0 \leq \min\{\alpha_i^-, \beta_i^-\} \leq \alpha_i^+ + \beta_i^+ \leq p_i^- - 1$, we may find $c_i > 0$ small enough so that

$$(c_i k_0)^{\alpha_i^+ + \beta_i^+} \geq c_i^{p_i^- - 1}.$$

Let $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$. It holds

$$\begin{aligned} m_i \int_\Omega \underline{u}_1^{\alpha_i(x)} \underline{u}_2^{\beta_i(x)} \varphi_i \, \mathrm{d}x &= m_i c_i^{\alpha_i^+ + \beta_i^+} \int_\Omega \zeta_1^{\alpha_i(x)} \zeta_2^{\beta_i(x)} \varphi_i \, \mathrm{d}x \\ &\geq (c_i k_0)^{\alpha_i^+ + \beta_i^+} m_i \int_\Omega d(x)^{\alpha_i(x)+\beta_i(x)} \varphi_i \, \mathrm{d}x \\ &= (c_i k_0)^{\alpha_i^+ + \beta_i^+} \int_\Omega |\nabla \zeta_i|^{p_i(x)-2} \nabla \zeta_i \nabla \varphi_i \, dx \\ &\geq c_i^{p_i^- - 1} \int_\Omega |\nabla \zeta_i|^{p_i(x)-2} \nabla \zeta_i \nabla \varphi_i \, dx. \end{aligned} \tag{8}$$

Proof of Application 2 II

Besides, Theorem 2 ensures the existence of $x_i \in \Omega$ such that,

$$\int_\Omega |\nabla \underline{u}_i|^{p_i(x)-2}\nabla \underline{u}_i \nabla \varphi_i \; dx = \int_\Omega |\nabla (c_i\zeta_i)|^{p_i(x)-2}\nabla (c_i\zeta_i)\nabla \varphi_i \; dx$$

$$= c_i^{p_i(x_i)-1}\int_\Omega |\nabla \zeta_i|^{p_i(x)-2}\nabla \zeta_i \nabla \varphi_i \; dx \leq c_i^{p_i^- -1}\int_\Omega |\nabla \zeta_i|^{p_i(x)-2}\nabla \zeta_i \nabla \varphi_i \; dx$$

Thus, we infer that

$$\int_\Omega |\nabla \underline{u}_i|^{p_i(x)-2}\nabla \underline{u}_i \nabla \varphi_i \; \mathrm{d}x \leq m_i \int_\Omega \underline{u}_1^{\alpha_i(x)} \underline{u}_2^{\beta_i(x)} \varphi_i \; \mathrm{d}x,$$

Main hypotheses

(H1) $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$.

(H2) $p_i \in C^1(\overline{\Omega})$, $1 < p_i^- \leq p_i^+ < \infty$ $(i = 1, 2)$.

(H3) $f_i(x, u_1, u_2, \nabla u_1, \nabla u_2)$ $(i = 1, 2)$ is of Carathéodory type.

(H4) There exists constants $m_i, M_i > 0$ with functions $\alpha_i, \beta_i, \gamma_i, \overline{\gamma}_i \in C(\overline{\Omega})$, s.t.
$m_i s_1^{\alpha_i(x)} s_2^{\beta_i(x)} \leq f_i(x, s_1, s_2, \xi_1, \xi_2) \leq M_i \left(s_1^{\alpha_i(x)} s_2^{\beta_i(x)} + |\xi_1|^{\gamma_i(x)} + |\xi_2|^{\overline{\gamma}_i(x)} \right)$
$(i = 1, 2)$, for a.e. $x \in \Omega$ and for all $s_1, s_2 > 0$.

(H5) $0 \leq \min\{\alpha_i^-, \beta_i^-\} \leq \alpha_i^+ + \beta_i^+ \leq p_i^- - 1$
and $0 \leq \min\{\gamma_i^-, \overline{\gamma}_i^-\} \leq \max\{\gamma_i^+, \overline{\gamma}_i^+\} \leq p_i^- - 1$.

Exisence theorem

Theorem

Under assumptions (H), system (S) has at least one positive nontrivial solution in $C_0^{1,\nu}(\overline{\Omega}) \times C_0^{1,\nu}(\overline{\Omega})$ for certain $\nu \in (0,1)$.

For each $(z_1, z_2) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, we consider the auxiliary problem

$$(\mathrm{S}_{(z_1,z_2)}) \qquad \begin{cases} -\Delta_{p_i(x)} u_i = f_i(x, z_1, z_2, \nabla z_1, \nabla z_2) & \text{in } \Omega \\ u_i = 0 \quad \text{on } \partial\Omega, \quad i = 1, 2. \end{cases}$$

Now, we introduce the closed, bounded and convex set

$$\mathcal{K}_C = \left\{ (y_1, y_2) \in C_0^1(\overline{\Omega})^2 : \underline{u}_i \leq y_i \text{ in } \Omega \text{ and } \|\nabla y_i\|_\infty \leq C \right\}, \tag{9}$$

where $C > 0$ is a constant sufficiently large.

Proof of the existence theorem I

Define the map

$$\begin{array}{llll} \mathcal{T}: & \mathcal{K}_C & \to & C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}) \\ & (z_1, z_2) & \mapsto & \mathcal{T}(z_1, z_2) = (u_1, u_2)_{(z_1, z_2)}, \end{array}$$

where (u_1, u_2) is required to satisfy $(\mathrm{S}_{(z_1,z_2)})$. We prove that :

- $\mathcal{T}$ is well defined (Browder-Minty's theorem)
- $\mathcal{T}$ is compact (Fan's regularity theorem)
- $\mathcal{T}$ is continuous (Ascoli-Arzela's theorem)
- $\mathcal{T}(\mathcal{K}_C) \subset \mathcal{K}_C$ (Weak comparison principle)
- $\mathcal{T}$ has at least one fixed point (Schauder's fixed point theorem)

Multiplicity theorem

Theorem

Let Y be a nonempty closed convex subset of E and U be an open subset of Y. Assume that the application $\mathcal{F}$ is compact and differentiable over the set U, with $\mathcal{F}(0) = \mathcal{F}'(0) = 0$, where 0 refers to the trivial function of E, and for any $t \in [0,1]$, $0 \notin (I - t\mathcal{F})(\partial U)$. Also, suppose the set

$$\Gamma = \{x \in U \backslash \{0\} \,|\, x = \mathcal{F}(x) \text{ and } 1 \text{ is not an eigenvalue of } \mathcal{F}'(x)\} \tag{10}$$

is not empty. Then

1. *Γ is finite;*
2. *For any $x \in \Gamma$, x is isolated;*
3. *If card(Γ) is odd, problem $x = \mathcal{F}(x)$ possesses at least card$(\Gamma) + 1$ nontrivial solutions in U.*

Proof of the multiplicity theorem I

We prove the following points :

- For each $x \in \Gamma, x$ is isolated (Amann's theorem) (11)
- Γ is finite (since $\mathcal{F}(\Gamma) = \Gamma$ and $\mathcal{F}$ is compact) (12)
- $H_t = (1-t)I + t(I - \mathcal{F}) = I - t\mathcal{F}, \ t \in [0,1].$ (13)
- $i(\mathcal{F}, U, Y) = i(H_t, U, Y) = i(I, U, Y) = 1.$ (Amann's index fixed point) (14)
- $i(\mathcal{F}, U, Y) = i(\mathcal{F}, U_1, Y) + i(\mathcal{F}, U_2, Y),$ (15)
- $$i(\mathcal{F}, U, Y) = \sum_{x \in \Gamma \cup \{0\}} i(\mathcal{F}, B(x,\rho), Y) \quad (16)$$
  $$+ i(\mathcal{F}, U \backslash \left\{\cup_{x \in \Gamma \cup \{0\}} B(x,\rho)\right\}, Y). \quad (17)$$
- $$i(\mathcal{F}, B(0,\rho), Y) = d(I - \mathcal{F}, B(0,\rho), 0) = d(I - \mathcal{F}'(0), B(0,\rho), 0) \quad (18)$$
  $$= d(I, B(0,\rho), 0) = 1. \quad (19)$$

Proof of the multiplicity theorem II

- $$\sum_{x\in\Gamma} i(\mathcal{F}, B(x,\rho), Y) + i(\mathcal{F}, U\setminus\left\{\cup_{x\in\Gamma\cup\{0\}} B(x,\rho)\right\}, Y) = 0. \tag{20}$$
- $$i(\mathcal{F}, B(x,\rho), Y) = (-1)^m \text{ (Amann's theorem)} \tag{21}$$

[1] H. Amann, *Lectures on some fixed point theorem*, Conselho nacional de pesquisas, Instituto de Matemática Pura e Aplicada, 1975.

[2] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM review, 18(4), 620-709, 1976.

Application I

Consider the nonlinear Dirichlet system

$$\begin{cases} -\Delta_{p(x)}u = f_1(x,u,v,\nabla u,\nabla v) & \text{in } \Omega \\ -\Delta_{q(x)}v = f_2(x,u,v,\nabla u,\nabla v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \Leftrightarrow \begin{cases} L(u,v) = S(u,v) & \text{in } \Omega \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases} \tag{22}$$

If $(u_1^\star, u_2^\star) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ is a solution of (22) then $(\phi_1^\star, \psi_1^\star) = T(u_1^\star, u_2^\star) \in W^{-1,p'(x)}(\Omega) \times W^{-1,q'(x)}(\Omega)$ is a solution of the fixed point problem

$$\begin{cases} (\phi,\psi) = (S \circ T)(\phi,\psi) & \text{in } \Omega \\ (\phi,\psi) = (0,0) & \text{on } \partial\Omega, \end{cases}$$

where T is the inverse operator of L.

Application II

$$\mathcal{F} = S \circ T : L^\infty(\Omega)^2 \to L^\infty(\Omega)^2$$

We need to prove the following statements :

- $S \circ T$ is compact and differentiable over $B(0, C) \subset (L^\infty(\Omega)^2)$
- $(S \circ T)(0,0) = (S \circ T)'(0,0) = (0,0)$
- $(0,0) \notin (I - tS \circ T)(S(0, C))$
- $\Gamma = \{(u,v) \in B(0,C) \backslash \{(0,0)\} \mid (u,v) = S \circ T(u,v)$ and 1 is not an eigenvalue of $(S \circ T)'(u,v)\}$ is not empty

Additional hypotheses

(A1) $p_i \in C^1(\overline{\Omega})$, $2 \leq p_i^- \leq p_i^+ < \infty$ $(i = 1, 2)$.

(A2) There exists bounded functions $g : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \Omega \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ such that for any $(u_1, u_2), (\Phi_1, \Phi_2) \in X^{-1, p_1'(x), p_2'(x)}(\Omega)$,

$$S'(u_1, u_2)[\Phi_1, \Phi_2] = < g(x, u_1, u_2), (\Phi_1, \Phi_2) > + < h(x, \nabla u_1, \nabla u_2), (\nabla \Phi_1, \nabla \Phi_2) >$$

(A3) There exists $C > 1$ such that $C(\|g(., u_1, u_2)\|_\infty + \|h(., \nabla u_1, \nabla u_2)\|_\infty) < 1$

Typical example : $S(u, v) = (S_1(u, v), S_2(u, v))$ with

$$S_i(u, v) = c_i(x) u^{\alpha_i(x)} v^{\beta_i(x)} + d_i(x) |\nabla u|^{\gamma_i(x)} + e_i(x) |\nabla v|^{\overline{\gamma}_i(x)}$$

Existence theorem

Theorem

Under assumptions (H) and (A), system (S) has at least two different non-trivial solutions in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$.

[1] L. Li, *Coexistence theorems of steady states for predator-prey interacting systems*, Transactions of the American Mathematical Society, 305(1), 143-166, 1988.

Step 1 : $S \circ T$ is compact and differentiable over $B(0, C)$

$(u_1, u_2) = T(z_1, z_2)$ if and only if (u_1, u_2) is the solution of the problem

$$\begin{cases} -\Delta_{p_1(x)} u_1 = z_1 & \text{in } \Omega \\ -\Delta_{p_2(x)} u_2 = z_2 & \text{in } \Omega \\ (u_1, u_2) = (0, 0) & \text{on } \partial\Omega. \end{cases} \tag{23}$$

- Existence : $(u_1, u_2) \in C_0^{1,\alpha}(\Omega)^2$ $(\alpha \in (0, 1))$ satisfying (23).
- Compact embedding : $C_0^{1,\alpha}(\Omega)^2 \hookrightarrow C_0^1(\Omega)^2$
- Inverse compacity : $T : L^\infty(\Omega)^2 \to C_0^1(\Omega)^2$ is compact.
- Continuity : $S : C_0^1(\Omega)^2 \to L^\infty(\Omega)^2$ is continuous.

Step 2 : $(S\circ T)(0,0)=(S\circ T)'(0,0)=(0,0)$ I

$(S\circ T)(0,0)=(0,0)\Leftrightarrow(0,0)$ is a solution of

$$\begin{cases} -\Delta_{p_i(x)}u_i=f_i(x,u_1,u_2,\nabla u_1,\nabla u_2) & \text{in } \Omega\\ u_i=0 & \text{on } \partial\Omega \end{cases}$$

Since $L\circ T(g_1,g_2)=(g_1,g_2)$, it follows for any $(\Phi_1,\Phi_2)\in L^2(\Omega)^2$ that

$$\begin{aligned}(L\circ T)'(g_1,g_2)[\Phi_1,\Phi_2]&=L'(T(g_1,g_2))[T'(g_1,g_2)[\Phi_1,\Phi_2]]\\ &=L'(u_1,u_2)(T'(g_1,g_2)[\Phi_1,\Phi_2])=(\Phi_1,\Phi_2),\end{aligned}\tag{24}$$

Step 2 : $(S \circ T)(0,0) = (S \circ T)'(0,0) = (0,0)$ II

- $L'(u_1,u_2)(V_1,V_2) = \Big(-div((p_1(x)-1)|\nabla u|^{p_1(x)-2}\nabla V_1), -div((p_2(x)-1)|\nabla v|^{p_2(x)-2}\nabla V_2)\Big)$
- $T'(g_1,g_2)[\Phi_1,\Phi_2] = (L'(u_1,u_2))^{-1}(\Phi_1,\Phi_2),$
- $(S \circ T)'(g_1,g_2)[\Phi_1,\Phi_2] = S'(u_1,u_2)[T'(g_1,g_2)[\Phi_1,\Phi_2]] = (0,0)$

Step 3 : $(0,0) \notin (I - tS \circ T)(S(0,C))$

Let $(z_1, z_2) \in B(0,C)$, and for any $t \in (0,1)$ $(u_1, u_2) = \mathcal{T}_t(z_1, z_2)$, that is

$$\begin{cases} -\Delta_{p_i(x)} u_i = t f_i(x, z_1, z_2, \nabla z_1, \nabla z_2) & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases} \Leftrightarrow L(u_1, u_2) = tS(z_1, z_2).$$

Then

$$\|u_i\|_{1,\tau} \leq \overline{k}_{p_i} \|f_i(., z_1, z_2, \nabla z_1, \nabla z_2)\|_\infty^{\frac{1}{p_i^{\pm}-1}} \leq \overline{k}_{p_i} M_i \left(C^{\alpha_i^+ + \beta_i^+} + C^{\gamma_i^+} + C^{\overline{\gamma}_i^+} \right)^{\frac{1}{p_i^- - 1}} < C,$$

for C sufficiently large. It follows that for any $(z_1, z_2) \in S(0,C)$

$$L(u_1, u_2) \neq tS(z_1, z_2) \Leftrightarrow (I - tS \circ T)(z_1, z_2) \neq (0,0).$$

Step 4 : Γ is not empty I

Lemma

Under assumptions (H) and (A), let $(u_1, u_2) \in C_0^1(\Omega)^2$ *satisfying*

$$C(\|g(., u_1, u_2)\|_\infty + \|h(., \nabla u_1, \nabla u_2)\|_\infty) < 1 \tag{25}$$

where g, h *are defined in (A2) and* $C > 1$ *is a constant large enough. Consider the set*

$$E = \left\{ (V_1, V_2) \in H_0^1(\Omega)^2 \,|\, 1 = \sum_{i=1,2} \int_\Omega |\nabla u_i|^{p_i(x)-2} |\nabla V_i|^2 dx, \text{ and } \sum_{i=1,2} \int_\Omega |\nabla V_i|^2 dx \le C \right\},$$

where $C > 0$ *is the same constant as in (25). Then*

$$\inf_{(V_1,V_2)\in E} \left\{ \int_\Omega < (\Phi_1, \Phi_2) - (S \circ T)'(g_1, g_2)[\Phi_1, \Phi_2], (V_1, V_2) > dx \right\} > 0,$$

Step 4 : Γ is not empty II

Proposition

Under assumptions (H) and (A), let $(u_1, u_2) \in C_0^1(\Omega)^2$ *satisfying (25), and* $(g_1, g_2) = L(u_1, u_2)$. *Then* 1 *is not an eigenvalue of* $(S \circ T)'(g_1, g_2)$.

[1] L. Li, *Coexistence theorems of steady states for predator-prey interacting systems*, Transactions of the American Mathematical Society, 305(1), 143-166, 1988.

Step 4 : Γ is not empty III

- $(I-(S\circ T)'(g_1,g_2))(\phi_1,\psi_1)=\lambda_1(I-(S\circ T)'(g_1,g_2))(\phi_1,\psi_1)>(0,0)$

- $(I+P(1,1)-h(.,u_1,u_2))(\phi_1,\psi_1)>((S\circ T)'(u_1,u_2)+P(1,1)-h(.,u_1,u_2))\,(\phi_1,\psi_1)$
for any constant $P>\|g(.,u_1,u_2)\|_\infty$

- $T'(g_1,g_2):L^2(\Omega)^2\to L^2(\Omega)^2$ is compact, linear and positive

- $(S\circ T)'(g_1,g_2)[\Phi_1,\Phi_2]+P(1,1)(V_1,V_2)-h(.,\nabla u_1,\nabla u_2)(\nabla V_1,\nabla V_2)$ is compact, linear and positive

- $r\left[(I+P(1,1)-h(.,u_1,u_2))^{-1}\mathcal{A}(u_1,u_2)\right]<1$ (Li's theorem)

Thank You