Méthodes spectrales et preuves assistées par ordinateur pour l'étude d'EDP semi-linéaires sur $\mathbb{R}^m$

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Computer-assisted proofs in dynamics

- Cesari [1964] Functional analysis and Galerkin's method
- Lanford [1982] Feigenbaum conjectures
- Fefferman & de la Llave [1986] Relativistic stability of matter
- Mischaikow & Mrozek [1995] Chaos in the Lorenz equations
- ⋮
- Jaquette & van den Berg [2019] Wright's conjecture (delay eq.)
- Wilczak & Zgliczyński [2020] Chaos in Kuramoto-Sivashinsky
- van den Berg, Breden, L. & van Veen [2021] Spontaneous periodic orbits in the Navier-Stokes equations
- Buckmaster, Cao-Labora & Gómez-Serrano [2022] Imploding solutions for 3D compressible fluids
- Chen & Hou [2022] Blowup in 3D Euler's equations

What about PDEs defined on unbounded domains?

A motivation: develop a method to prove **constructively** existence of localized planar patterns, which arise in the study of

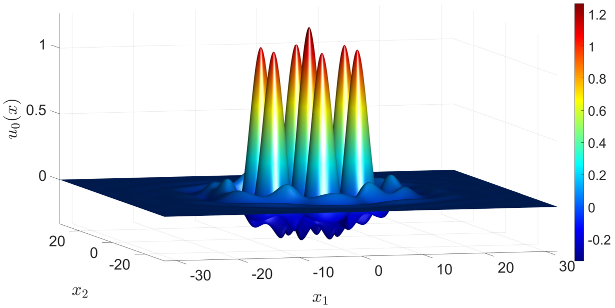
- dryland vegetation
- nonlinear optics
- phase-field crystals
- water waves
- neural field equations
- granular dynamics
- binary fluid convection & ferrofluids

A mathematical challenge

Despite the prevalence and importance of localized planar patterns, little is known about them from a mathematical perspective.

A motivating example on unbounded domains: non-radial localized stationary patterns in the Swift-Hohenberg PDE

$$(1 + \Delta)^2 u + \mu u + \nu_1 u^2 + \nu_2 u^3 = 0.$$



- D. Hill, J. Bramburger & D. Lloyd. *Approximate localised dihedral patterns near a Turing instability*. Nonlinearity, 36(5):2567, 2023.

Formulation of the problem

We look for stationary planar solutions to

$$u_t = (1+\Delta)^2 u + \mu u + \nu_1 u^2 + \nu_2 u^3 = 0$$

with $u \to 0$ as $|x| \to \infty$. Assume that $\mu > 0$.
Let

$$\begin{aligned} \mathbb{L} &\overset{\text{def}}{=} \mu I_d + (I_d + \Delta)^2 \\ \mathbb{G}(u) &\overset{\text{def}}{=} \nu_1 u^2 + \nu_2 u^3 \\ \mathbb{F}(u) &\overset{\text{def}}{=} \mathbb{L}u + \mathbb{G}(u) \end{aligned}$$

then equivalently we have the following zero finding problem

$$\mathbb{F}(u) = 0 \quad \text{and} \quad u \to 0 \text{ as } |x| \to \infty.$$

Main steps to get the computer-assisted proofs

- Develop a method (fully) based on Fourier series.
- Treat the problem on $\mathbb{R}^2$ as a periodic problem (on a big enough square).
- Construct an approximate solution u_0 on $\mathbb{R}^2$ using Fourier series.
- Construct an approximate inverse for $D\mathbb{F}(u_0)$ using Fourier series.
- Verify rigorously the hypotheses of a Newton-Kantorovich theorem close to u_0 to show existence of an exact solution to $\mathbb{F}(u) = 0$

Operator representations

Let $\Omega_0 \stackrel{\text{def}}{=} (-d,d)^2$, the operators defined in the previous slides have a Fourier series representation on Ω_0.

Operator	Fourier transform	Fourier series
$\mathbb{L} = \mu I_d + (I_d + \Delta)^2$	$l(\xi) = \mu + (1 - \lvert\xi\rvert^2)^2$	$L_p = \text{diag}\left(l(\frac{\pi}{d}n)\right)$
u^2	$\hat{u} * \hat{u}$	$U * U$
$\mathbb{F}(u) = \mathbb{L}u + \mathbb{G}(u)$	$l(\xi)\hat{u}(\xi) + \widehat{\mathbb{G}(u)}$	$F_p(U) = L_p U + G_p(U)$

The Hilbert spaces H^l and X^l

First ingredient: a natural Hilbert space for the solutions.

- **Assume** $l(\xi) \overset{\text{def}}{=} \mathcal{F}(\mathbb{L})(\xi) = \mu + (1 - |\xi|^2)^2 > 0$
- Define $H^l \overset{\text{def}}{=} \{u \in L^2, \;\; \|u\|_l < \infty\}$, where

$$\|u\|_l^2 \overset{\text{def}}{=} \|\mathbb{L}u\|_2^2 = \int_{\mathbb{R}^2} l(\xi)^2 |\hat{u}(\xi)|^2 d\xi$$

- H^l is a Banach algebra if $\frac{1}{l} \in L^2 \implies \mathbb{G} : H^l \to L^2$ is well-defined
- Similarly,

$$X^l \overset{\text{def}}{=} \{U \in \ell^2, \;\; \|U\|_l < \infty\}$$

where $\|U\|_l = \|L_p U\|_2$.

Elimination of the translation invariance

- There is a natural translation invariance of the set of solutions.
- Restrict to D_2 symmetric functions to eliminate the invariance.
- $H_e \overset{\text{def}}{=} \{u \in H^l : u(x_1, x_2) = u(-x_1, x_2) = u(x_1, -x_2), \forall x_1, x_2 \in \mathbb{R}^2\}$
- L_e is the restriction of L^2 to D_2 symmetric functions.

The zero finding problem becomes

$$\mathbb{F}(u) = \mathbb{L}u + \mathbb{G}(u) = 0 \quad \text{and} \quad u \in H_e$$

and $\mathbb{F} : H_e \to L_e$.

Newton-Kantorovich approach

- Suppose we have an approximate solution $u_0 \in H_e$
- Suppose that $D\mathbb{F}(u_0)^{-1} : L_e \to H_e$ is a bounded linear operator

Prove that there exists $r > 0$ such that $\mathbb{T} : \overline{B_r(u_0)} \to \overline{B_r(u_0)}$

$$\mathbb{T}(u) \stackrel{\text{def}}{=} u - D\mathbb{F}(u_0)^{-1}\mathbb{F}(u),$$

is a contraction $\implies$ there exists a unique solution in $\overline{B_r(u_0)} \subset H_e$

Newton-Kantorovich approach

A Newton-Kantorovich type theorem

Consider u_0 a numerical approximation of $\mathbb{F}(u) = 0$ and suppose that $\mathbb{A} : L_e \to H_e$ is an approximate inverse of $D\mathbb{F}(u_0)$. Consider bounds $\mathcal{Y}_0$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$ satisfying

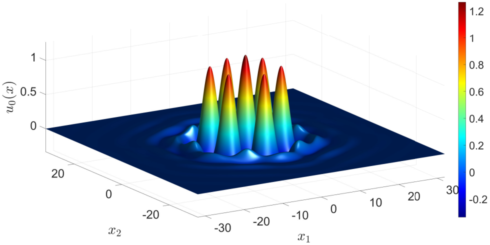
- $\|\mathbb{A}\mathbb{F}(u_0)\|_l \leq \mathcal{Y}_0$
- $\|I_d - \mathbb{A}D\mathbb{F}(u_0)\|_l \leq \mathcal{Z}_1$
- $\|\mathbb{A}\left(D\mathbb{F}(u_0) - D\mathbb{F}(v)\right)\|_l \leq \mathcal{Z}_2(r)r,$ for all $v \in \overline{B_r(u_0)}$.

If there exists $r > 0$ such that

$$\mathcal{Z}_1 < 1 \quad \text{and} \quad 2\mathcal{Y}_0\mathcal{Z}_2(r) < (1 - \mathcal{Z}_1)^2$$

then $\mathbb{T} : \overline{B_r(u_0)} \to \overline{B_r(u_0)}$ is a contraction.

Computation of the numerical approximation u_0



- $\text{supp}(u_0) \subset \Omega_0 \stackrel{\text{def}}{=} (-d, d)^2$.
- Numerically, we can define u_0 on Ω_0 by a Fourier series $U_0 = (a_n)_{n \in \mathbb{Z}^2}$ where $a_n = 0$ if $|n|_\infty > N$ for some $N \in \mathbb{N}$.
- u_0 is fully determined by U_0: $u_0(x) = \mathbf{1}_{\Omega_0}(x) \displaystyle\sum_{|n|_\infty \le N} a_n e^{i \frac{\pi}{d} n \cdot x}$.

Computation of the numerical approximation u_0

- Assume that using a numerical scheme (e.g. Newton's method), we computed $\tilde{U}_0$ such that $F_p(\tilde{U}_0) \approx 0$ (Fourier problem on Ω_0).
- We build Γ such that Γ^* is full rank and such that the Fourier sequence

$$U_0 \stackrel{\text{def}}{=} \tilde{U}_0 - \Gamma^*(\Gamma\Gamma^*)^{-1}\Gamma\tilde{U}_0$$

  corresponds to a function $(u_0)_{\Omega_0} \in H_0^4(\Omega_0)$
- Choose u_0 as the zero extension of the function $(u_0)_{\Omega_0} \in H_0^4(\Omega_0)$.
- Hence, $u_0 \in H^4(\mathbb{R}^2) = H^l$.

Construction of an approximate inverse $\mathbb{A}$ for $D\mathbb{F}(u_0)$

- Approximate inverse $\mathbb{A} : L_e \to H_e$ of $D\mathbb{F}(u_0) = \mathbb{L} + D\mathbb{G}(u_0)$
- $\mathbb{A} = \mathbb{L}^{-1}\mathbb{B}$ where $\mathbb{B} : L_e \to L_e$ approximates the inverse of $I_d + D\mathbb{G}(u_0)\mathbb{L}^{-1}$
- Choose $\mathbb{B} = I_d + \mathbb{B}_{\Omega_0}$ where $\mathbb{B}_{\Omega_0} = \mathbf{1}_{\Omega_0}\mathbb{B}_{\Omega_0}\mathbf{1}_{\Omega_0}(\star)$

Fourier series representation of $\mathbb{B}_{\Omega_0}$

If $\mathbb{B}_{\Omega_0} : L_e \to L_e$ is a bounded linear operator such that $(\star)$ holds, then $\mathbb{B}_{\Omega_0}$ has a unique Fourier series representation $B_p = (b_{i,j})_{i,j\in\mathbb{Z}^2} : \ell^2_e \to \ell^2_e$ given by

$$b_{p,j} \stackrel{\text{def}}{=} \frac{1}{|\Omega_0|}\int_{\Omega_0} (\mathbb{B}_{\Omega_0}\psi_j)(x)\overline{\psi_i(x)}dx$$

where $\psi_n(x) \stackrel{\text{def}}{=} e^{i\frac{\pi}{d}n\cdot x}$ for all $n \in \mathbb{Z}^2$ and all $x \in \Omega_0$.

Construction of an approximate inverse $\mathbb{A}$ for $D\mathbb{F}(u_0)$

- Numerically, we build B_p such that $I_d + B_p \approx \left(I_d + DG_p(U_0)L_p^{-1}\right)^{-1}$
- B_p has a representation $\mathcal{F}^{-1}(B_p) = \mathbb{B}_{\Omega_0} : L_e \to L_e$ and $\mathbb{B}_{\Omega_0} = \mathbf{1}_{\Omega_0}\mathbb{B}_{\Omega_0}\mathbf{1}_{\Omega_0}$
- Build $\mathbb{B} = I_d + \mathbb{B}_{\Omega_0} : L_e \to L_e$
- Build $\mathbb{A} = \mathbb{L}^{-1}\mathbb{B} : L_e \to H_e$
- Denote $A_p = L_p^{-1}(I_d + B_p) \approx DF_p(U_0)^{-1}$

The bound $\mathcal{Y}_0$

- $\|\mathbb{A}\mathbb{F}(u_0)\|_l = \|\mathbb{L}\mathbb{A}\mathbb{F}(u_0)\|_2 = \|\mathbb{B}\mathbb{F}(u_0)\|_2$
- $\mathbb{F}(u_0)$ has a Fourier series representation $F_p(U_0)$
- Using Parseval's identity,
  $\|\mathbb{B}\mathbb{F}(u_0)\|_2^2 = |\Omega_0| \|(I_d + B_p)F_p(U_0)\|_2^2 = |\Omega_0| \|A_p F_p(U_0)\|_l^2$
- Hence, the rigorous upper bound $\mathcal{Y}_0$ is obtained via Fourier series and interval arithmetic computations.

The bound $\mathcal{Z}_1$

We notice that

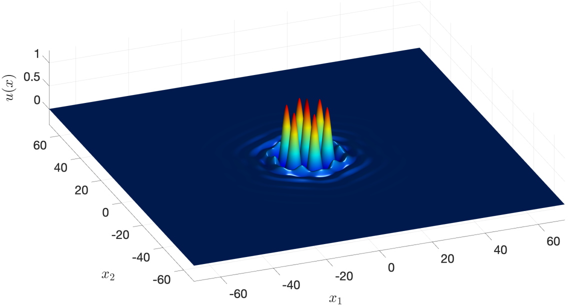
- $\|I_d - \mathbb{A}D\mathbb{F}(u_0)\|_l \leq \|I_d - A_p DF_p(U_0)\|_l + \mathcal{Z}_{1,1}$.
- Note that $\mathcal{Z}_{1,1} = O(e^{-ad})$, which comes from the fact that

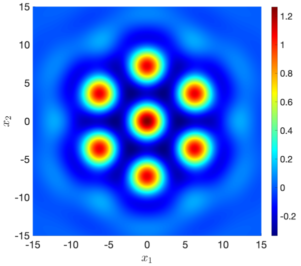
  $$|\mathcal{F}^{-1}(\frac{1}{l})(x)| \leq Ce^{-a|x|}$$

  where a is the smallest imaginary part of the roots of l.
- $\|I_d - A_p DF_p(U_0)\|_l \leq Z_1$ is computed with Fourier series computations.

Constructive proofs of existence of localized patterns in SH

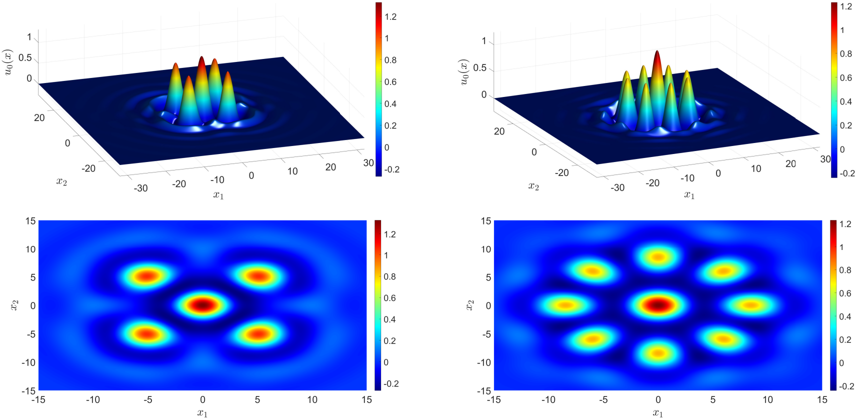
$$(1+\Delta)^2 u + \mu u + \nu_1 u^2 + \nu_2 u^3 = 0.$$





- Parameter values: $\mu = 0.32$, $\nu_1 = -1.6$ and $\nu_2 = 1$
- Size of the Fourier projection: $N = 130$, Domain size: $d = 70$ (vectors of size $(N+1)^2 = 17161$)
- Bounds: $\mathcal{Y}_0 = 2.7 \times 10^{-5}$, $\mathcal{Z}_1 = 0.02$, $\mathcal{Z}_2 = 212$
- $r = 2.8 \times 10^{-5}$ (rigorous global error bound in H^l)

Constructive proofs of existence of localized patterns in SH



Rigorous interval arithmetic computations using Julia

julia

JuliaIntervals
Guaranteed computations

O. Hénot
McGill

RadiiPolynomial.jl

References

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2. Dan J. Hill, Jason J. Bramburger and David J.B. Lloyd. *Approximate localised dihedral patterns near a Turing instability*. Nonlinearity, 36(5):2567, 2023.
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