High frequency analysis of the Dirichlet to Neumann and of the impedance operator for an infinite cylinder (and an infinite elliptic cylinder) coated with dielectric material

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Problem studied

Define $\Omega = \mathcal{O} \times \mathbf{R}$, $\mathcal{O}$ bounded convex openset in $\mathbf{R}^2$. Let $\mathcal{O}^e$ bounded convex openset in $\mathbf{R}^2$ such that $\mathcal{O} \subset \mathcal{O}^e$ and there exists $\alpha_0 > 0$ such that $\text{dist}(\mathcal{O}, (\mathcal{O}^e)^c) \geq \alpha_0$.
Assume that $\mathcal{O}$ is a perfectly conducting body, and that $\mathcal{O}_e$ is filled with a dielectric material of dielectric constants ϵ, μ satisfying $\Im \epsilon\mu < 0$, $\Re \epsilon\mu > 0$, fixed (independent of ω).
Our aim is to replace the Helmholtz equation in $\mathcal{O} \cup (\mathcal{O}_e - \mathcal{O})$ by a boundary condition on $\partial\mathcal{O}_e$.

Abstract classical result

Theorem

The problem

$$\begin{cases} (\Delta + \omega^2 \epsilon \mu) u = 0, (\mathcal{O}^e - \mathcal{O}) \times \mathbf{R}, \\ u|_{\partial \mathcal{O} \times \mathbf{R}} = 0, \\ u|_{\partial \mathcal{O}^e \times \mathbf{R}} = u_0 \end{cases} \tag{1}$$

has a unique solution $U(u_0)$ *in* $H^1((\mathcal{O}^e - \mathcal{O}) \times \mathbf{R})$ *for each* $u_0 \in H^{\frac{1}{2}}(\partial \mathcal{O}^e \times \mathbf{R})$.
The application $u_0 \to \partial_n U(u_0)$, ∂_n *being the normal derivative on* $\partial \mathcal{O}^e \times \mathbf{R}$ *is well defined from* $H^{\frac{1}{2}}(\partial \mathcal{O}^e \times \mathbf{R})$ *to* $H^{-\frac{1}{2}}(\partial \mathcal{O}^e \times \mathbf{R})$.
It is called the Dirichlet to Neumann operator (DtN).

The Dirichlet to Neumann operator for the infinite cylinder

Assume $\mathcal{O} = B(0, r_0)$, $\mathcal{O}^e = B(0, R)$, $R > r_0$. Fourier transform in z (k). Equation

$$\begin{cases} (\Delta + \epsilon\mu\omega^2 - k^2)\hat{u}(., k) = 0, r_0 < r < R, \\ \hat{u}(., k)|_{r=r_0} = 0, \hat{u}(., k)|_{r=R} = \hat{u}_0(., k). \end{cases}$$

Fourier series expansion $\hat{u}(x, y, k) = \sum_{n=0}^{+\infty} e^{in\theta} v_n(r, k)$, where v_n solves the Bessel equation:

$$\frac{1}{r}\frac{d}{dr}(r\frac{dv_n}{dr}) + (\epsilon\mu\omega^2 - k^2 - \frac{n^2}{r^2})v_n = 0.$$

Dirichlet condition $v_n(r_0) = 0$.

The solution in $C(r_0, R)$ is, necessarily, of the form $v_n(r) = A(J_n(k_3 r)Y_n(k_3 r_0) - Y_n(k_3 r)J_n(k_3 r_0))$, $k_3 = \sqrt{\epsilon\mu\omega^2 - k^2}$, $\Im k_3 < 0$.

Expression of the Dirichlet to Neumann multiplier

From u having a Fourier transform,
$k \to A(J_n(k_3 r)Y_n(k_3 r_0) - Y_n(k_3 r)J_n(k_3 r_0))$ belongs to $\mathcal{S}'(\mathbf{R})$.
Decay condition on A.
From $\hat{u}_0(.,k) = \sum \hat{u}_0(n,k)e^{in\theta}$, $\sum_n |\hat{u}_0(n,k)|^2(1+n^2)^{\frac{1}{2}} < +\infty$.

$$v_n(r) = \frac{J_n(k_3 r)Y_n(k_3 r_0) - Y_n(k_3 r)J_n(k_3 r_0)}{J_n(k_3 R)Y_n(k_3 r_0) - Y_n(k_3 R)J_n(k_3 r_0)}\hat{u}_0(n,k) \qquad (2)$$

Proposition

There exists a constant C, independent of (k, ω, n), such that for all n, k, $|v_n(r)| \leq C|\hat{u}_0(n,k)|$.

$$v_n'(R) = k_3 \frac{J_n'(k_3 R)Y_n(k_3 r_0) - Y_n'(k_3 R)J_n(k_3 r_0)}{J_n(k_3 R)Y_n(k_3 r_0) - Y_n(k_3 R)J_n(k_3 r_0)} v_n(R)$$

$DtN(u)(\theta) = \sum_{n=-\infty}^{+\infty} DtN_n.u_n.e^{in\theta}$ *for* $u \in H^{\frac{1}{2}}(\partial\mathcal{O}^e)$

Usual asymptotics

Use $H_n^{(1)}(k_3r) = J_n(k_3r) + iY_n(k_3r)$. Analysis with

$$\begin{aligned} H_n^{(1)}(k_3r) &= \tfrac{2}{\pi k_3 r}[P^*(n,k_3r) + iQ^*(n,k_3r)]e^{i(k_3r-(\frac{1}{2}n+\frac{\pi}{4}))} \\ (H^{(1)})'_n(k_3r) &= \tfrac{2}{\pi k_3 r}[iR^*(n,k_3r) - S^*(n,k_3r)]e^{i(k_3r-(\frac{1}{2}n+\frac{\pi}{4}))} \end{aligned}.$$

Leading order term related to $H^{(1)}$ thanks to $|e^{i(k_3R-(\frac{1}{2}n+\frac{\pi}{4})-k_3r_0+(\frac{1}{2}n+\frac{\pi}{4}))}| = e^{-\Im k_3(R-r_0)}$.

$$P^* = 1 + O((k_3r)^{-2}), Q^* = \tfrac{4n^2-1}{8k_3r}(1 + O((k_3r)^{-2})),$$

$$R^* = 1 + O((k_3r)^{-2}), S^* = \tfrac{4n^2+3}{8k_3r}(1 + O((k_3r)^{-2})).$$

Estimate: $\frac{v_n'(R)}{v_n(R)} = ik_3(1 + \frac{i}{2k_3R})(1 + O((k_3R)^{-1}))$.

$$DtN(n)\text{: } ik_3 - \tfrac{1}{2R} + \dots.$$

Unsatisfactory: microlocal analysis would lead to

$$i\sqrt{\epsilon\mu\omega^2 - k^2 - \frac{n^2}{R^2}}. \tag{3}$$

(approximation $i(k_3 - \frac{n^2}{2k_3R^2})$). How come?

Bessel function toolbox: the Debye expansions in the hyperbolic zone (I)

Assumption: $\frac{n}{R}$ is a high frequency parameter.

'Hyperbolic 'regime ($\Re\omega^2\epsilon\mu - k^2 > \frac{n^2}{R^2}$). Introduce β such that $k_3 R = \frac{n}{\cos\beta}$. Phase $\Psi(n,\beta) = n(\tan\beta - \beta)$.

$$\begin{cases} H_n^{(1)}(\frac{n}{\cos\beta}) = \sqrt{\frac{2}{\pi n\tan\beta}}(L - iM)e^{i\Psi(n,\beta)}, \\ (H_n^{(1)})'(\frac{n}{\cos\beta}) = \sqrt{\frac{\sin 2\beta}{\pi n}}(iN - O)e^{i\Psi(n,\beta)}, \end{cases}$$

$$L = 1 + O(n^{-2}), n\tan\beta M = \frac{1}{8} + \frac{5}{24}\tan^{-2}\beta + O(n^{-2}),$$
$$N = 1 + O(n^{-2}), n\tan\beta O = \frac{3}{8} + \frac{7}{24}\tan^{-2}\beta + O(n^{-2}).$$

$$k_3 = \frac{d}{dr}(\frac{n}{\cos\beta}) = \frac{n\sin\beta}{\cos^2\beta}\frac{d\beta}{dr}$$

$\rightarrow (\frac{d\Psi}{dr})^2 = k_3^2\sin^2\beta = \epsilon\mu\omega^2 - k^2 - \frac{n^2}{r^2}$. Eikonal equation.

Bessel function toolbox the Debye expansions in the elliptic zone (II)

'Elliptic' regime: $\Re\omega^2\epsilon\mu - k^2 < \frac{n^2}{R^2}$, define z through $k_3 r = nz$.
$H_n^{(1)}(nz) = 2e^{-i\frac{\pi}{3}}(\frac{\zeta}{1-z^2})^{\frac{1}{4}}[n^{-\frac{1}{3}}Ai(jn^{\frac{2}{3}}\zeta)A - n^{-\frac{5}{3}}j\zeta^{-\frac{1}{2}}BAi'(jn^{\frac{2}{3}}\zeta)]$,
$A = 1 + O(n^{-2})$, $B = U_1 + O(n^{-2})$, $\zeta = (\frac{3}{2}\int_z^1 \frac{\sqrt{1-t^2}}{t}dt)^{\frac{2}{3}}$
Equivalent:

$$Ai(jn^{\frac{2}{3}}\zeta) \simeq (jn^{\frac{2}{3}}\zeta)^{-\frac{1}{4}}e^{-\frac{2}{3}(jn^{\frac{2}{3}}\zeta)^{\frac{3}{2}}} = j^{-\frac{1}{4}}n^{-\frac{1}{6}}\zeta^{-\frac{1}{4}}e^{n\int_z^1 \frac{\sqrt{1-t^2}}{t}dt}.$$

In the elliptic regime $(nz = k_3 r)$*, same eikonal equation:*

$$\tilde{\Psi}(z) = -\frac{2}{3}in\zeta^{\frac{3}{2}} \to \frac{d\tilde{\Psi}}{dr} = -in\frac{\sqrt{1-z^2}}{z}\frac{dz}{dr} \to (\frac{d\tilde{\Psi}}{dr})^2 = (1 - \frac{1}{z^2})(n\frac{dz}{dr})^2.$$

The global asymptotics

Dominant Bessel function $H^{(1)}$. Expansion

$$k_3 \frac{\sqrt{\frac{\sin 2\beta}{\pi n}}(iN-O)}{\sqrt{\frac{2}{\pi n \tan\beta}}(L-iM)} = ik_3\sqrt{\frac{2\sin^2\beta}{2}}\frac{N+iO}{L-iM} = i\frac{d\Psi}{dr}\frac{N+iO}{L-iM}$$
$$= ik_3 \sin\beta(1 + i\frac{1}{n\tan\beta}(\frac{4}{8} + \tan^{-2}\beta\frac{5+7}{24}))$$

$$DtN_n = i\sqrt{\epsilon\mu\omega^2 - k^2 - \frac{n^2}{R^2}} - \frac{1}{2R}\frac{\epsilon\mu\omega^2 - k^2}{\epsilon\mu\omega^2 - k^2 - \frac{n^2}{R^2}} + \ldots \quad (4)$$

In the elliptic regime, let $q_* := n^{\frac{1}{3}}\zeta_*^{\frac{1}{2}}\frac{Ai}{Ai'}(n^{\frac{2}{3}}\zeta_*)$, $q_* \simeq -1$.

$$DtN_n = -k_3\frac{2}{z_*}\sqrt{\frac{1-z_*^2}{4\zeta_*}}\frac{Ai'}{Ai}(n^{\frac{2}{3}}\zeta_*)\frac{D_*+C_*(q_*n)^{-1}}{A_*+B_*q_*n^{-1}}, \text{ that is}$$

$$DtN_n = \sqrt{k^2 + \frac{n^2}{R^2} - \epsilon\mu\omega^2} + \frac{1}{2R}\frac{\epsilon\mu\omega^2 - k^2}{k^2 + \frac{n^2}{R^2} - \epsilon\mu\omega^2} + \ldots \quad (5)$$

The sphere and Maxwell's equations

Solutions of the Maxwell's equations
$\nabla \wedge \mathbf{E} = i\omega\mu\mathbf{H}, \nabla \wedge \mathbf{H} = -\frac{i\omega}{c^2\mu}\mathbf{E}$,
with $k^2c^2 = \omega^2$, $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$:

$$\mathbf{E} = \sum_{m,n} A_{m,n} j_n(kr) V_{m,n} + C_{m,n} [\frac{Y_{m,n} n(n+1) j_n(kr)}{r} \vec{e}_r + (k\partial_x j_n(kr) + \frac{j_n(kr)}{r}) V^{\perp}_{m,n}]$$

(and the equivalent formulation with the solution y_n). Relations:

$$\nabla \wedge (j_n(kr) V_{m,n}) = \frac{Y_{m,n} n(n+1) j_n(kr)}{r} \vec{e}_r + (k\partial_x j_n(kr) + \frac{j_n(kr)}{r}) V^{\perp}_{m,n},$$

$$\nabla \wedge (\frac{Y_{m,n} n(n+1) j_n(kr)}{r} \vec{e}_r + (k\partial_x j_n(kr) + \frac{j_n(kr)}{r}) V^{\perp}_{m,n}) = k^2 j_n(kr) V_{m,n} \vec{e}_r.$$

The solution with given tangent traces on the coated layer

Introduce

$$\tau_1^n(r) = \frac{j_n(kr)y_n(kr_0)-y_n(kr)j_n(kr_0)}{j_n(kR)y_n(kr_0)-y_n(kR)j_n(kr_0)},$$
$$\tau_3^n(r) = \frac{j_n(kr)y_n'(kr_0)-y_n(kr)j_n'(kr_0)}{j_n'(kR)y_n'(kr_0)-y_n'(kR)j_n'(kr_0)},$$
$$\tau_2^n(r) = \frac{j_n'(kr)y_n(kr_0)-y_n'(kr)j_n(kr_0)}{j_n(kR)y_n(kr_0)-y_n(kR)j_n(kr_0)},$$
$$\tau_4^n(r) = \frac{j_n'(kr)y_n'(kr_0)-y_n'(kr)j_n'(kr_0)}{j_n'(kR)y_n'(kr_0)-y_n'(kR)j_n'(kr_0)}.$$

Put $\vec{e}_r \wedge \mathbf{E}|_{r=r_0} = 0$, $\vec{e}_r \wedge \mathbf{E}|_{r=R} = \sum_{m,n}[E^1_{m,n}V^\perp_{m,n} - E^2_{m,n}V_{m,n}]$.

Get $\mathbf{E} = \sum_{m,n} E^1_{m,n}\tau^1_n(r)V_{m,n} + E^2_{m,n}(\frac{n(n+1)\tau^3_n(r)Y_{m,n}}{r}\vec{e}_r + \frac{k\tau^4_n(r)+\frac{\tau^3_n(r)}{r}}{k+R^{-1}\tau^3_n(R)}V^\perp_{m,n})$.

Deduce $\nabla \wedge \mathbf{E} =$

$\sum_{m,n} E^1_{m,n}[\frac{n(n+1)\tau^1_n(r)Y_{m,n}}{r}\vec{e}_r + (k\tau^2_n(r) + \frac{\tau^1_n(r)}{r})V^\perp_{m,n}] + E^2_{m,n}\frac{k^2\tau^3_n(r)}{k+R^{-1}\tau^3_n(R)}V_{m,n}$.

The impedance operator

Define Z the operator such that

$$\vec{e}_r \wedge \mathbf{E}|_{r=R} = Z[\vec{e}_r \wedge \vec{e}_r \wedge \mathbf{H}|_{r=R}].$$

It exists and is diagonal in the spherical harmonics basis, thanks to $\nabla \wedge \mathbf{E} = i\omega\mu\mathbf{H}$, of expression

$$Z^1_{m,n} = -i\sqrt{\frac{\mu}{\epsilon}}(\frac{1}{\tau^3_n(R)} + \frac{1}{kR}), Z^2_{m,n} = i\sqrt{\frac{\mu}{\epsilon}}\frac{1}{\tau^2_n(R) + \frac{1}{kR}}$$

thanks to $i\omega\mu(\vec{e}_r \wedge \vec{e}_r \wedge \mathbf{H}) = -i\omega\mu(H^1_{m,n}V_{m,n} + H^2_{m,n}V^{\perp}_{m,n})$.

Define $\cos\beta = \frac{n+\frac{1}{2}}{kR}$.

$$Z^1_{m,n} = -i\sqrt{\frac{\mu}{\epsilon}}(i\sin\beta\frac{L-iM}{N+iO}+\frac{1}{2kR}) = \sqrt{\frac{\mu}{\epsilon}}\sin\beta T^1_{m,n}, T^1_{m,n} = 1+\frac{i}{2kR}(\frac{\cos\beta}{\sin^2\beta}-1)$$

$$Z^2_{m,n} = i\sqrt{\frac{\mu}{\epsilon}}\frac{1}{i\sin\beta\frac{L-iM}{N+iO} + \frac{1}{2kR}} = \sqrt{\frac{\mu}{\epsilon}}\frac{1}{\sin\beta T^2_{m,n}}, T^2_{m,n} = 1 + \frac{i}{2kR}(\frac{\cos\beta}{\sin^2\beta} - 1)$$

Helmholtz equation in elliptic coordinates

$\mathcal{O} = \{\rho(\cosh u \cos v, \sinh u \sin v), u \in [0, u_0], v \in [0, 2\pi)\}$,
$\mathcal{O}^e = \{\rho(\cosh u \cos v, \sinh u \sin v), u \in [0, u_1], v \in [0, 2\pi)\}$
Helmholtz operator

$$\Delta = \frac{1}{\rho^2(\cosh^2 u - \cos^2 v)}(\partial^2_{u^2} + \partial^2_{v^2}) + \frac{\partial^2}{\partial z^2}.$$

Separation of variables $u(x, y, z) = e^{ikz}F(u)G(v)$

$$F''(u) + (\frac{k_3^2}{2}\rho^2 \cosh^2 u - a)F = 0, u_0 \leq u \leq u_1,$$

(Modified Mathieu)

$$G''(v) - (\frac{k_3^2}{2}\rho^2 \cos^2 v - a)G = 0, 0 \leq v \leq 2\pi.$$

(Mathieu)

Bloch theory, Modified Mathieu functions

Periodic solutions of the Mathieu equation $\Rightarrow$ Floquet modes.
Yields $a_n(k_3\rho)$ (even solutions), $b_n(k_3\rho)$ (odd solutions).
Equation

$$F''(u) = \left(\begin{matrix} a_n(k_3\rho) \\ b_n(k_3\rho) \end{matrix} - \frac{k_3^2}{2}\rho^2 \cosh^2 u \right) F(u)$$

Construction of all solutions in $[u_0, u_1]$ as

$$\frac{F_g(u)F_d(u_0) - F_d(u)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

BOUNDED.

$$DtN \simeq \frac{F_g'(u_1)F_d(u_0) - F_d'(u_1)F_g(u_0)}{F_g(u_1)F_d(u_0) - F_d(u_1)F_g(u_0)}.$$

Obtention of the DtN

Normal vector:

$$\vec{n} = \frac{1}{\sqrt{\sinh^2 u_1 \cos^2 v + \cosh^2 u_1 \sin^2 v}} (\sinh u_1 \cos v, \cosh u_1 \sin v).$$

Normal derivative involves u **AND** v.
Curvature at a point of the boundary $\partial \mathcal{O}^e$

$$\rho(s(v)) = \frac{\rho}{\cosh u_1 \sinh u_1} (\cosh^2 u_1 \cos^2 v + \sinh^2 u_1 \sin^2 v)^{\frac{3}{2}}$$

The v parts couples modes.
Regimes for obtaining the values of the DtN:

- bottom of the well (near $-\frac{k_3^2 \rho^2}{2}$): classical Schrodinger well, (corresponds to n finite)
- above the maximum of the potential (elliptic regime)
- between the maximum and the minimum of the potential (hyperbolic region).

Asymptotic expressions used in the hyperbolic regime

Introduce $N = \sqrt{a_n(k_3\rho)}$, $\frac{d\Theta}{du} = \sqrt{\frac{2\rho^2 k_3^2}{a_n(k_3\rho)}\cosh^2 u - 1}$.
$F''(u) = (a_n(k_3\rho) - \frac{k_3^2\rho^2}{2}\cosh^2 u)F(u)$ rewrites
$F''(u) = (iN)^2(\Theta'(u))^2 F(u)$.

$$F(u) = y(\Theta(u)) \Rightarrow \frac{d\Theta}{du}\frac{d}{d\Theta}(\frac{d\Theta}{du}\frac{dy}{d\Theta}) = (iN)^2(\Theta'(u))^2 y(\Theta)$$

$r(\Theta) = (\Theta'(u))^{-\frac{1}{2}}y(\Theta) \Rightarrow \frac{d^2r}{d\Theta^2} = [(iN)^2 + \delta(\Theta)]r(\Theta) \Rightarrow r(\Theta) \simeq e^{\pm iN\Theta}(1 + O(N^{-2}))$.

Note that

$F'(u) = \Theta'(u)\frac{dy}{d\Theta}$, $\frac{dr}{d\Theta} = \frac{d}{d\Theta}((\Theta'(u))^{-\frac{1}{2}})y(\Theta) + (\Theta'(u))^{-\frac{1}{2}}\frac{dy}{d\Theta}$.

The mode N

Choose $\Im N\Theta(u_1) < 0$. Growing solution $e^{iN\Theta(u)}(1+O(N^{-2}))$.
DtN in the u part:

$$\begin{aligned}\frac{\frac{d}{du}(e^{iN\Theta(u)}(1+O(N^{-2})))}{e^{iN\Theta(u)}(1+O(N^{-2}))} &\simeq iN\Theta'(u)+O(N^{-1})\\ &= i\omega\rho\sqrt{\frac{\epsilon\mu-\eta^2}{2}\cosh 2u_1 - \frac{a_n(k_3\rho)}{\omega^2\rho^2}}+O(N^{-1}).\end{aligned}$$

No zeroth correction term.

Note that one can apply the same method for the Bessel functions: Whittaker equation on $w(r)=r^{\frac{1}{2}}u(r)$:
$w''=(k^2+\frac{n^2-\frac{1}{4}}{r^2}-\epsilon\mu\omega^2)w$, define $\varphi'(r)=\sqrt{\eta^2+\frac{n^2-\frac{1}{4}}{r^2\omega^2}-\epsilon\mu}$,
and write with the complex change of variable $(\varphi')^{\frac{1}{2}}\tilde{Y}(\varphi)=w(r)$ which yields $\tilde{Y}''=[(i\omega)^2+\delta(\varphi)]\tilde{Y}$.

Concluding remarks

1. Need to consider n as a high frequency parameter to have the total high frequency analysis of the DtN operator: $i\omega\sqrt{\epsilon\mu - \eta^2 - \tau^2}$, $k = \omega\eta$, $n = R\omega\tau$.
2. Never have, in the hypothesis $\Im\epsilon\mu$ independent of ω, the glancing regime
3. Need to include in the impedance operator the elliptic and the hyperbolic regime
4. Allows to get well defined solutions in the $\mathcal{S}'$ sense (not all solutions can have a Fourier transform)
5. In the case of an elliptic boundary, use the Floquet values a_n, b_n for the Mathieu equation
6. The impedance (DtN) operator couples all modes in the case of an elliptic boundary. No longer a Fourier multiplier.