Algorithms for coupling two turbulent incompressible fluids by a non linear interface law

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Geometry

$\mathbf{u}_i = \mathbf{u}_i(\mathbf{x}_h, z) = (\mathbf{u}_{i,h}, w_i)$, $\quad \mathbf{u}_{i,h} = (u_{i,x}, u_{i,y})$: velocity of fluid $i = 1, 2$, $\mathbf{x}_h = (x, y)$

$p_i = p_i(\mathbf{x}_h, z)$: pressure of fluid i,

Γ_1 top of fluid 1, Γ_2 bottom of fluid 2, Γ_{Int} : interface between both fluids

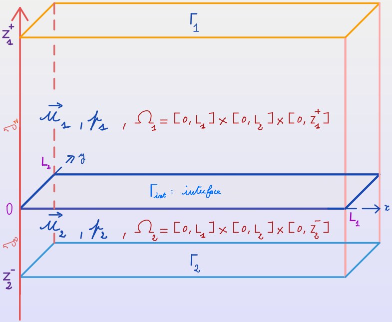

Figure – Computational box

Equations

The equations are the following.

$$\begin{cases} (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i - \nabla \cdot (\nu_t^i \nabla \mathbf{u}_i) + \nabla p_i = \mathbf{f}_i & \text{in } \Omega_i, \\ \nabla \cdot \mathbf{u}_i = 0, & \text{in } \Omega_i, \\ \nu_t^i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -C_D(\mathbf{u}_{i,h} - \mathbf{u}_{j,h})|\mathbf{u}_{i,h} - \mathbf{u}_{j,h}|, & \text{on } \Gamma_{Int}, \\ \nu_t^i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -c_i(\mathbf{u}_{i,h} - \mathbb{V}_i) & \text{on } \Gamma_i, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma_{Int} \cup \Gamma_i, \end{cases} \tag{1}$$

where $\mathbf{x}_h = (x, y) \in \mathbb{T}_2 = \mathbb{T}_2 = \frac{[0,L_1]\times[0,L_2]}{\mathbb{Z}^2}$,

$$\nu_t^i = \nu_t^i(z, \mathbf{u}_{i,h}) = \nu_i + C_s^i z^2 |\nabla \mathbf{u}_{i,h}|$$

is the eddy viscosity, ν_i the molecular viscosity, C_s^i the Smagorinsky's constant.

C_D and c_i are friction coefficients.

$\mathbb{V}_i$ velocities of the respective contact layers.

Functional spaces

Recall the the interface Γ_{Int} is given by

$$\Gamma_{Int} = \{(\mathbf{x}_h, 0), \mathbf{x}_h \in \mathbb{T}_2\}.$$

The boundaries Γ_i are given by

$$\Gamma_1 = \{(\mathbf{x}_h, z_1^+), \mathbf{x}_h \in \mathbb{T}_2\},$$

the top of fluid 1,

$$\Gamma_2 = \{(\mathbf{x}_h, z_2^-), \mathbf{x}_h \in \mathbb{T}_2\},$$

The bottom of fluid 2. For the simplicity we set

$$J_1 = [0, z_1^+], \quad J_2 = [z_2^-, 0],$$

where $z_1^+ > 0$ and $z_2^- < 0$. In other word, the domains Ω_i can be defined

$$\Omega_i = \mathbb{T}_2 \times J_i,$$

although for pratical calculations

$$\Omega_i = [0, L_1] \times [0, L_2] \times J_i.$$

Functional spaces

Let

$$W_i = \{\mathbf{u} \in H^1(\mathbb{T}_2 \times J_i),\ \mathbf{u}\cdot\mathbf{n}_i|_{\Gamma_{Int}\cap\Gamma_i} = 0\},$$

equipped with the norm

$$||\mathbf{u}||_{i,1} = ||\nabla\mathbf{u}||_{L^2(\mathbb{T}_2\times J_i)} + ||\mathbf{tru}||_{L^2(\Gamma_i)},$$

where $\mathbf{u} \to \mathbf{tru}$ denotes the trace operator, which will not systematically mentionned.
Let W denotes the product space

$$W = W_1 \times W_2.$$

Remark

In view of a mixed formulation velocity-pressure, we dot consider spaces for velocities with zero divergence.

Pressures will be seek in :

$$X = L^2(\mathbb{T}_2 \times J_1)/\mathbb{R} \times L^2(\mathbb{T}_2 \times J_2)/\mathbb{R}.$$

Diffusion :

$$A(\mathbf{U},\mathbf{V}) = \int_{\mathbb{T}_1\times J_1} \nu_t^1 \nabla \mathbf{u}_1 \cdot \nabla \mathbf{v}_1 + \int_{\mathbb{T}_2\times J_2} \nu_t^2 \nabla \mathbf{u}_2 \cdot \nabla \mathbf{v}_2.$$

We denote by a the continous operator $W \to W'$ given by :

$$\langle a(\mathbf{U}),\mathbf{V}\rangle = A(\mathbf{U},\mathbf{V}),$$

which is a monotone operator.

Transport :

$$B(\mathbf{U},\mathbf{V},\mathbf{W}) = B_1(\mathbf{u}_1,\mathbf{v}_1,\mathbf{w}_1) + B_2(\mathbf{u}_2,\mathbf{v}_2,\mathbf{w}_2),$$

where

$$B_i(\mathbf{u}_i,\mathbf{v}_i,\mathbf{w}_i) = \frac{1}{2}\left(\int_{\mathbb{T}_2\times J_i}(\mathbf{u}_i\cdot\nabla)\mathbf{v}_i\cdot\mathbf{w}_i - \int_{\mathbb{T}_2\times J_i}(\mathbf{u}_i\cdot\nabla)\mathbf{w}_i\cdot\mathbf{v}_i.\right).$$

We also will consider $b : W \times W \to W'$ which satisfies

$$\langle b(\mathbf{U},\mathbf{V}),\mathbf{W}\rangle = B(\mathbf{U},\mathbf{V},\mathbf{W}).$$

Variational formulation

Pressure :

$$N(P,\mathbf{V}) = \langle n(p), \mathbf{V}\rangle = -\int_{\mathbb{T}_2\times J_1} p_1 \nabla\cdot\mathbf{v}_1 - \int_{\mathbb{T}_2\times J_2} p_2\nabla\cdot\mathbf{v}_2.$$

Friction terms :

$$\langle g(\mathbf{U},\mathbf{V}),\mathbf{W}\rangle = G(\mathbf{U},\mathbf{V},\mathbf{W}) = C_D\int_{\Gamma_{Int}} |\mathbf{u}_{i,h}-\mathbf{u}_{j,h}|(\mathbf{v}_{i,h}-\mathbf{v}_{j,h})\cdot(\mathbf{w}_{i,h}-\mathbf{w}_{j,h})$$

$$h(\mathbf{U}),\mathbf{V})\rangle = H(\mathbf{U},\mathbf{V}) = c_1\int_{\Gamma_1}(\mathbf{u}_{1,h}-\mathbb{V}_1)\cdot\mathbf{v}_{1,h}+c_2\int_{\Gamma_2}(\mathbf{u}_{2,h}-\mathbb{V}_2)\cdot\mathbf{v}_{2,h}$$

Source term :

$$\langle\mathbf{F},\mathbf{V}\rangle = \int_{\mathbb{T}_2\times J_1}\mathbf{f}_1\cdot\mathbf{v}_1 + \int_{\mathbb{T}_2\times J_2}\mathbf{f}_2\cdot\mathbf{v}_2$$

Remark

Notice that for all $\mathbf{V}, \mathbf{U} \in W$,

$$B(\mathbf{V}, \mathbf{U}, \mathbf{U}) = 0,$$

when $\nabla \cdot \mathbf{u}_i = 0$,

$$B_i(\mathbf{u}_i, \mathbf{u}_i, \mathbf{v}_i) = \int_{\mathbb{T}_2 \times J_i} (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i \cdot \mathbf{v}_i,$$

and for all $P \in X$,

$$N(P, \mathbf{U}) = 0.$$

Moreover, for all $\mathbf{U} \in W$,

$$G(\mathbf{U}, \mathbf{U}, \mathbf{U}) = C_D \int_{\Gamma_{Int}} |\mathbf{u}_1 - \mathbf{u}_2|^3 \geq 0,$$

Variational formulation

Definition

We say that $(\mathbf{U}, P) = [(\mathbf{u}_1, \mathbf{u}_2), (p_1, p_2)] \in W \times X$ is a weak solution to Problem (1) if :

$\forall (\mathbf{V}, Q) \in W \times X$,

$$\begin{cases} \underbrace{B(\mathbf{U}, \mathbf{U}, \mathbf{V})}_{transport} + \underbrace{A(\mathbf{U}, \mathbf{V})}_{diffusion} + \underbrace{H(\mathbf{U}, \mathbf{V})}_{top,bottom} + \underbrace{G(\mathbf{U}, \mathbf{U}, \mathbf{V})}_{interface} + \underbrace{N(P, \mathbf{V})}_{pressure} = \underbrace{\langle \mathbf{F}, \mathbf{V} \rangle}_{sources}, \\ \underbrace{(Q, \nabla \cdot \mathbf{U}) = 0}_{incompressibility}. \end{cases}$$

In other words, Problem (1) can be written as : $\mathbf{U} \in W$, and

$$\begin{cases} b(\mathbf{U}, \mathbf{U}) + a(\mathbf{U}) + h(\mathbf{U}) + g(\mathbf{U}, \mathbf{U}) + n(P) = \mathbf{F} \in W', \\ \nabla \cdot \mathbf{U} = 0 \quad \text{in } X, \end{cases}$$

Existence Result

Theorem

Problem (1) *has an unique solution* $(\mathbf{U}, P) \in W \times X$ *that satisfies satisfies the energy equality :*

$$\underbrace{A(\mathbf{U},\mathbf{U}) + H(\mathbf{U},\mathbf{U})}_{\text{yields the norm in } W} + \underbrace{G(\mathbf{U},\mathbf{U})}_{\geq 0} = \langle \mathbf{F}, \mathbf{U} \rangle,$$

in particular for $i = 1, 2$,

$$\mathbf{u}_i \in W^{1,3}(\Omega_i; z^2).$$

Numerical algorithms

Simple recurrence algorithm

$$b(\mathbf{U}_n, \mathbf{U}_{n+1}) + a(\mathbf{U}_{n+1}) + g(\mathbf{U}_n, \mathbf{U}_{n+1}) + h(\mathbf{U}) + n(P_{n+1}) = \mathbf{F}.$$

Double recurrence algorithm

Let

$$\langle \tilde{g}(\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \mathbf{V}), \mathbf{W} \rangle =$$

$$C_D \int_{\Gamma_{Int}} |\mathbf{u}_{i,h}^{(1)} - \mathbf{u}_{j,h}^{(1)}|^{1/2} |\mathbf{u}_{i,h}^{(2)} - \mathbf{u}_{j,h}^{(2)}|^{1/2} (\mathbf{v}_{i,h} - \mathbf{v}_{j,h}) \cdot (\mathbf{w}_{i,h} - \mathbf{w}_{j,h})$$

$$b(\mathbf{U}_n, \mathbf{U}_{n+1}) + a(\mathbf{U}_{n+1}) + \tilde{g}(\mathbf{U}_{n-1}, \mathbf{U}_n, \mathbf{U}_{n+1}) + h(\mathbf{U}) + n(P_{n+1}) = \mathbf{F}.$$

2D simulations with FreeFem

Parameters : $z_0 = 20, z_a = 100, \; L = 150$ and

$$C_D = 0.01,$$

$$\nu_{\rm ah} = 10, \, \nu_{\rm av} = 1, \quad \nu_{\rm oh} = 1, \, \nu_{\rm ov} = 0,1$$

Friction at the top of the atmospheric layer and the bottom of the ocean layer :

$$V_{top} = u_{\rm logtop} \text{ (log law)} \qquad G_{\rm bot} = 0.$$

Pression regularization

$$\nu_{pa} = 10^{-5} = \nu_{po}$$

Smagorinsky's contants

$$C_s^o = 0.01, \qquad C_s^a = 0.001$$

Relative error ($< 10^{-3}$) :

$$\delta U_n = \frac{||\mathbf{U}_{n+1} - \mathbf{U}_n||_{L^2}}{||\mathbf{U}_n||_{L^2}}$$

2D simulations with FreeFem