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11ème Biennale de la
Société des Mathématiques
Appliquées et Industrielles

Bore propagation in channels with sloping banks: numerical and asymptotic analysis

M. Ricchiuto

Centre Inria de l'Université de Bordeaux, Team CARDAMOM

https://team.inria.fr/cardamom/

Inria

du 22 au 26
Mai 2023

11ème Biennale de la Société des Mathématiques Appliquées et Industrielles

Bore propagation in channels with sloping banks: numerical and asymptotic analysis

Joint work with:

P. Bonneton (EPOC), **R. Chassagne** (LEGI/IRSTEA),

A.G. Filippini (BRGM/R3C), **M. Kazolea** (Inria/CARDAMOM)

Géosciences pour une Terre durable

du 22 au 26 Mai 2023

Intro

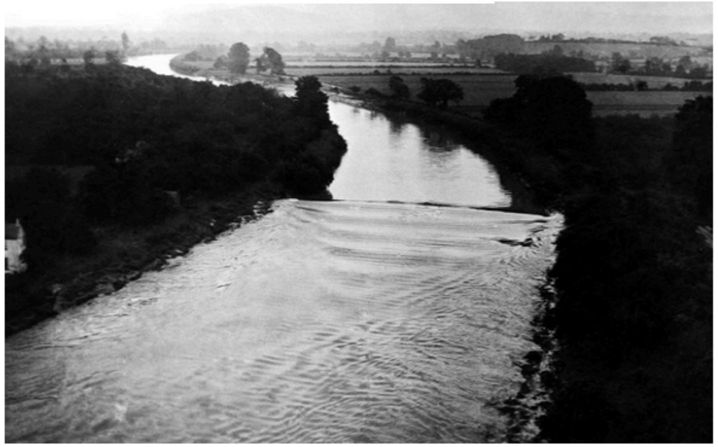
Bore propagation in channels with sloping banks: numerical and asymptotic analysis

?

Saint Pardon, Dordogne river

https://vimeo.com/106090912, Jean-Marc Chauvet, Septembre 2014

Severn River - England



Severn River - England

Gironde - France

Qiantang River - China

Amazon River - Brazil

Kampar River - Sumatra

Tidal bores bear striking similarity to tsunami bores, and bores generated in laboratory experiments

Naka river at Hitachinaka city, Japan
2011 Tohoku Tsunami

Sunaoshi River in Tagajo city, Japan
7.4 earthquake 21/11/2016

Sendai bay, Japan
2011 Tohoku Tsunami

Low Fr transition in Seine and Gironde: the unseen Mascaret

3 field campaigns :
a unique long-term high-frequency database

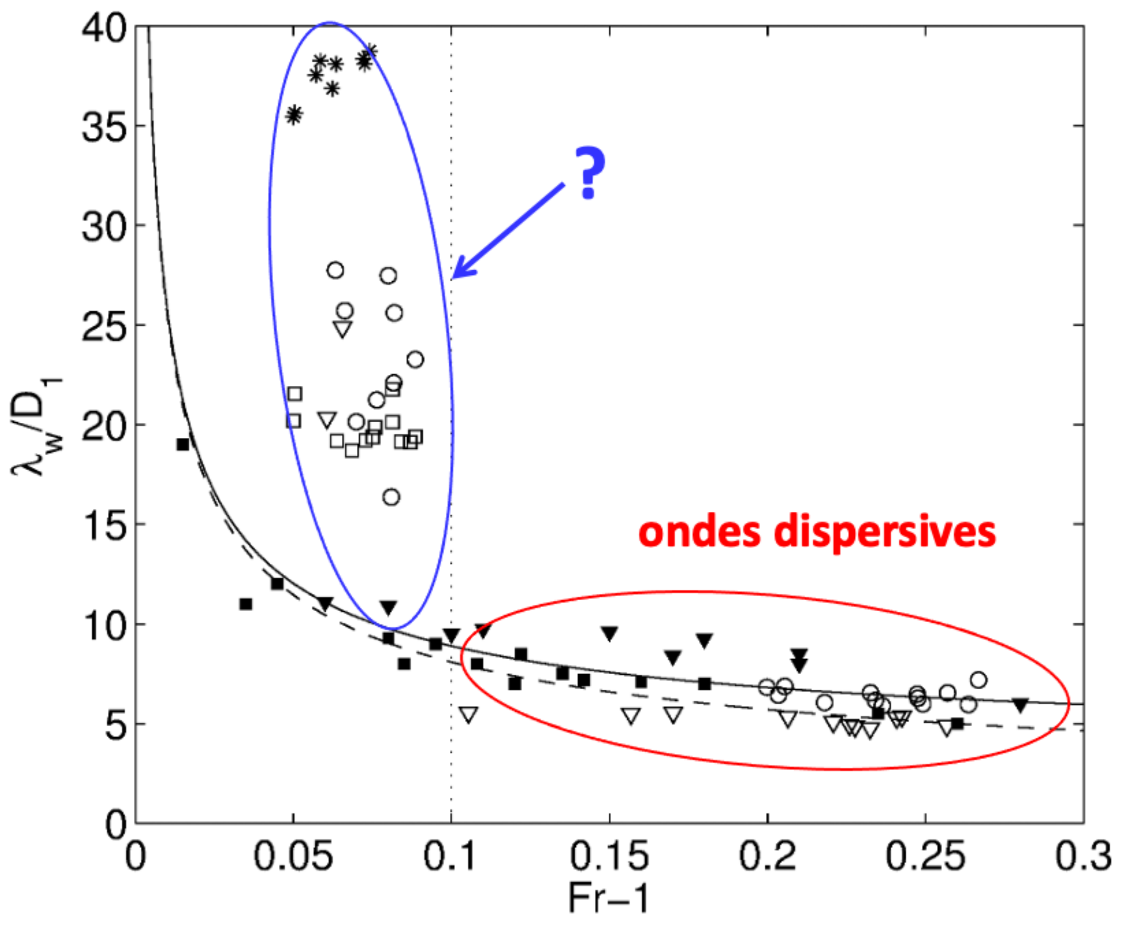


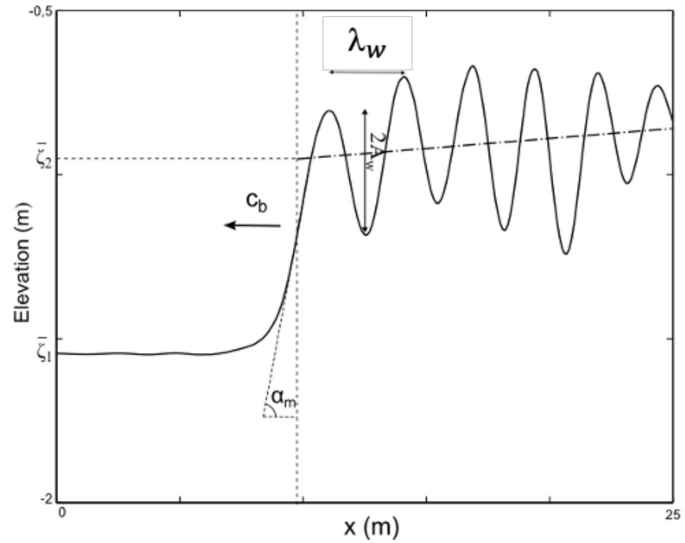




Bonneton et al, *Comptes Rendus Geoscience*, 2012

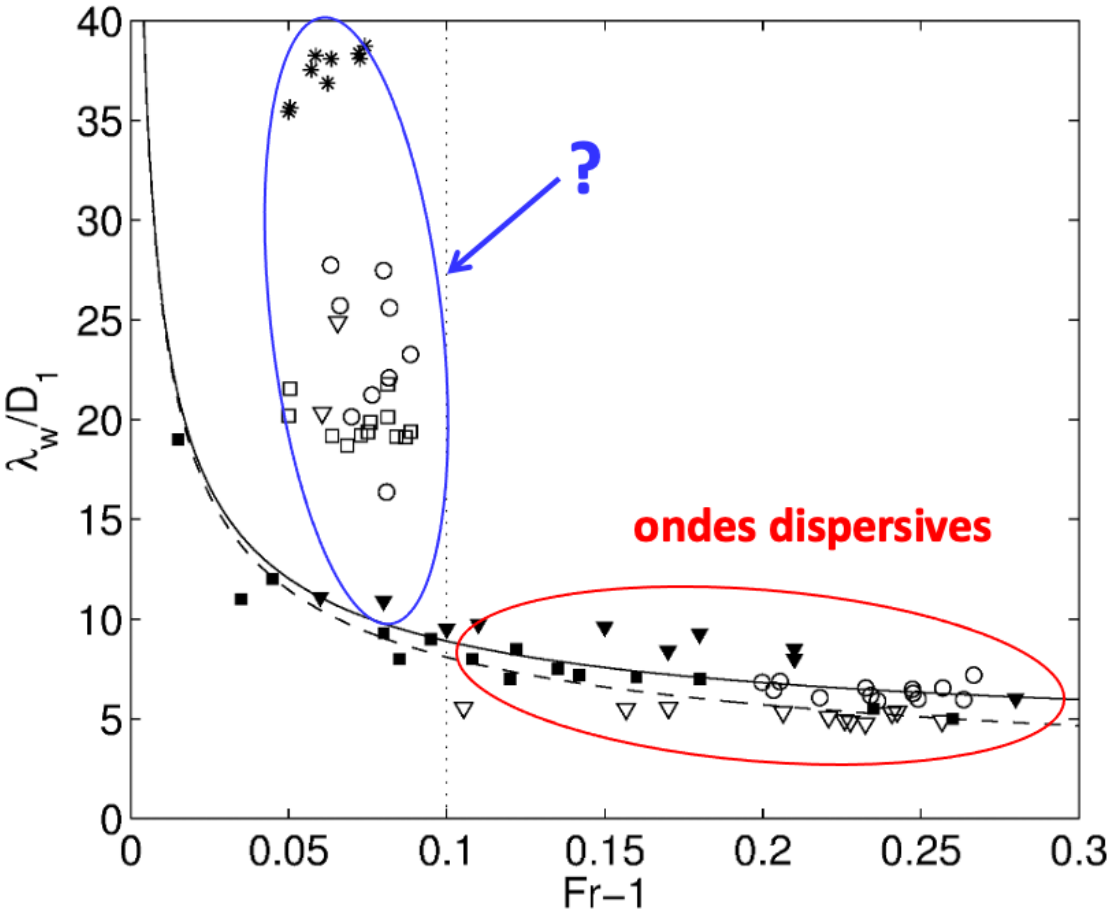
Bonneton et al, *J. Geophysical Research - Oceans,* 2015



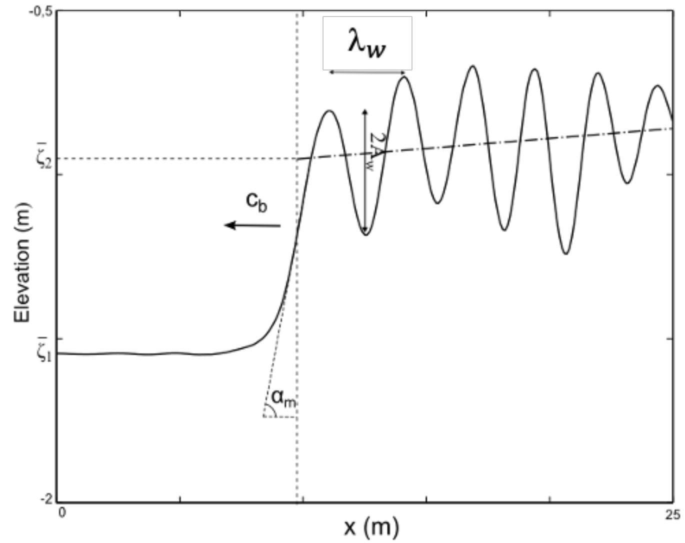


Transition for **Fr=F_T** (F_T≈1.1)

<u>*Ressaut de marée*</u> (n.lle terminologie **Bonneton et al**, *Comptes Rendus Geoscience*, 2012)
- not visible naked eyes
- mechanism not known



Transition for $Fr=F_T$ ($F_T \approx 1.1$)



Common undular tidal bore (mascaret): Favre wave (cf later)

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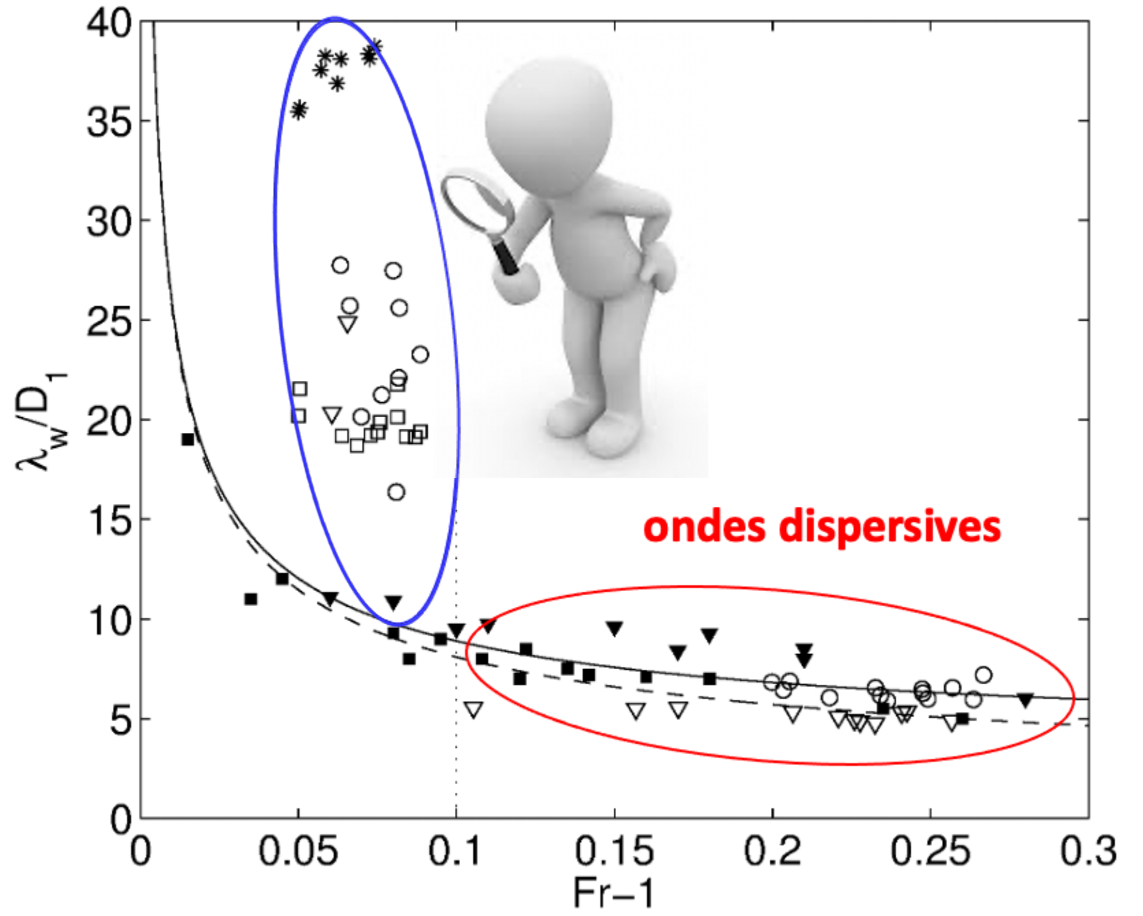

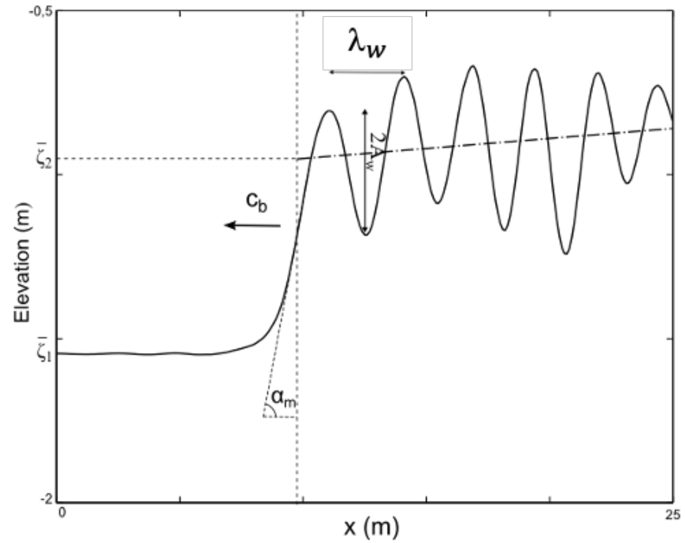

Common undular tidal bore (mascaret): Favre wave (cf later)

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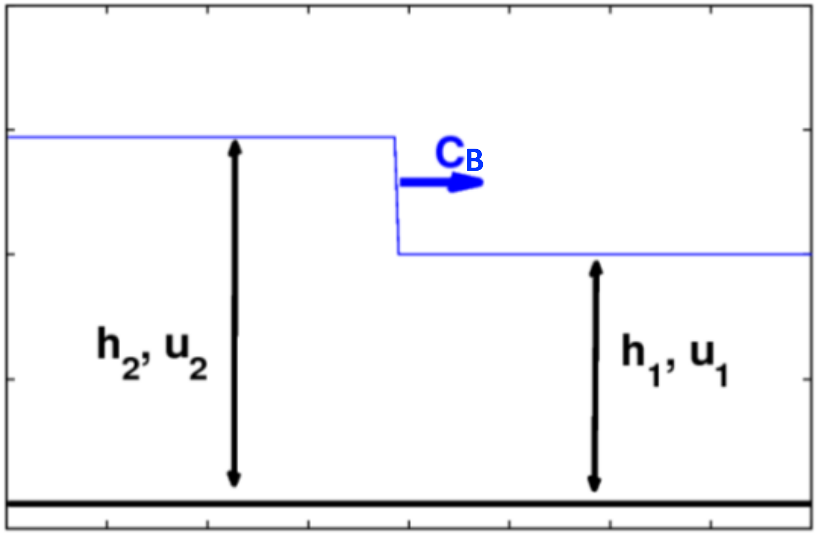
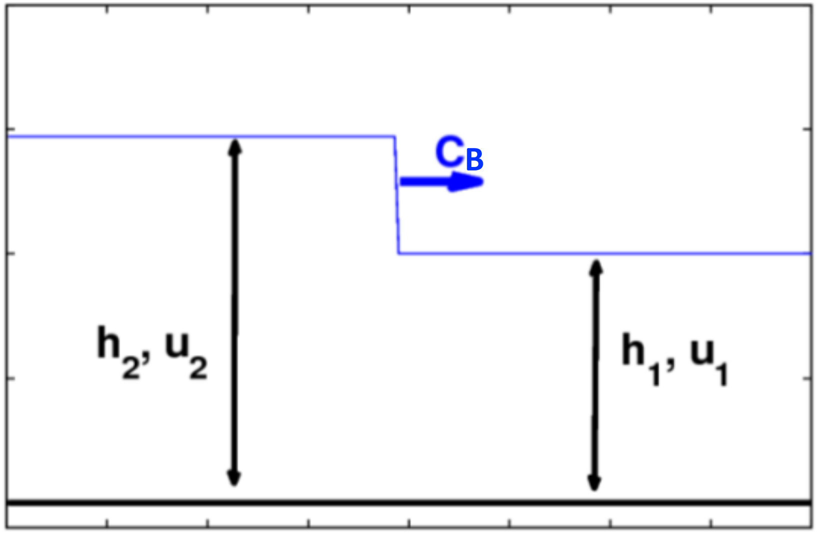
Nonlinearity vs dispersion

Bore: positive surge or hydraulic jump in translation



Bore: positive surge or hydraulic jump in translation



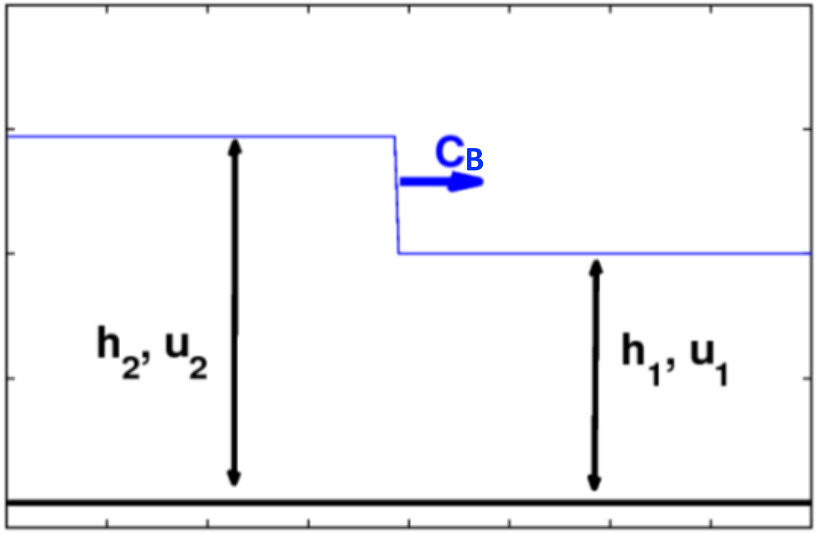
Shallow water equations (hydrostatic, shallow limit)

1.
2.
3.
4.

$$\partial_t h + \partial_x (hu) = 0$$

$$\partial_t (hu) + \partial_x (hu^2 + gh^2/2) = 0$$

Bore: positive surge or hydraulic jump in translation



Shallow water equations (hydrostatic, shallow limit)

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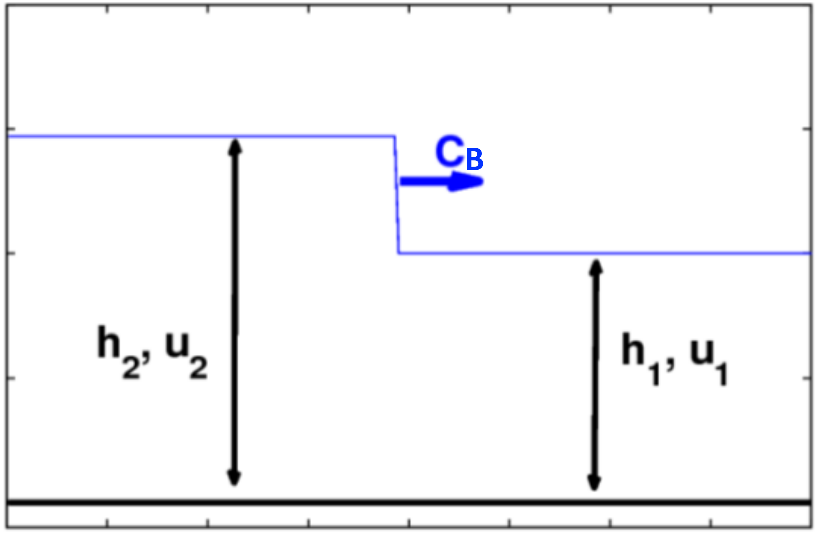
$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + A\,\partial_x \begin{pmatrix} h \\ hu \end{pmatrix} = 0$$

$$A \;=\; R\,\mathrm{diag}(u-c, u+c)R^{-1}\,, \quad c = \sqrt{gh}$$

Bore: positive surge or hydraulic jump in translation



Shallow water equations (hydrostatic, shallow limit)

1. hyperbolic
2. mathematical entropy (energy)
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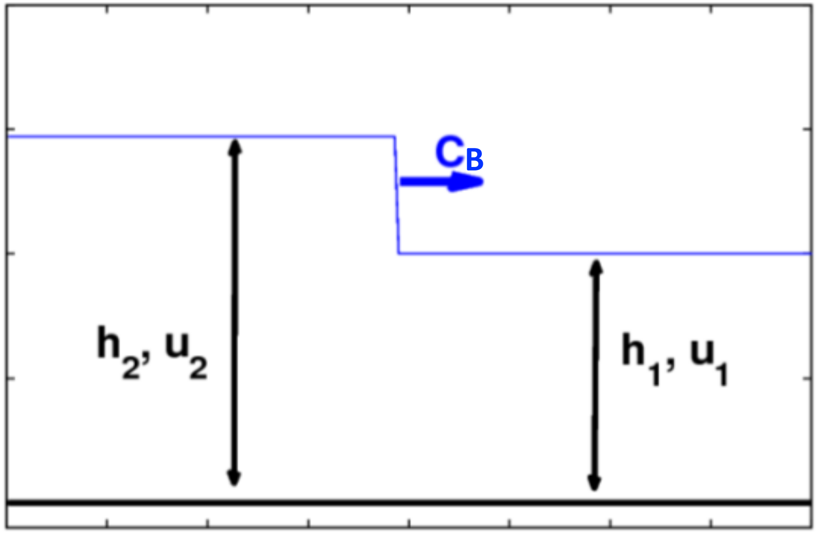
4.

$$\partial_t \underbrace{\left(gh^2/2 + hu^2/2\right)}_{E} + \partial_x \underbrace{\left[hu(gh + u^2/2)\right]}_{F_E} \leq 0$$

= 0 only for fully continuous flows/solutions (cf. later)

Bore: positive surge or hydraulic jump in translation



Shallow water equations (hydrostatic, shallow limit)

1. hyperbolic
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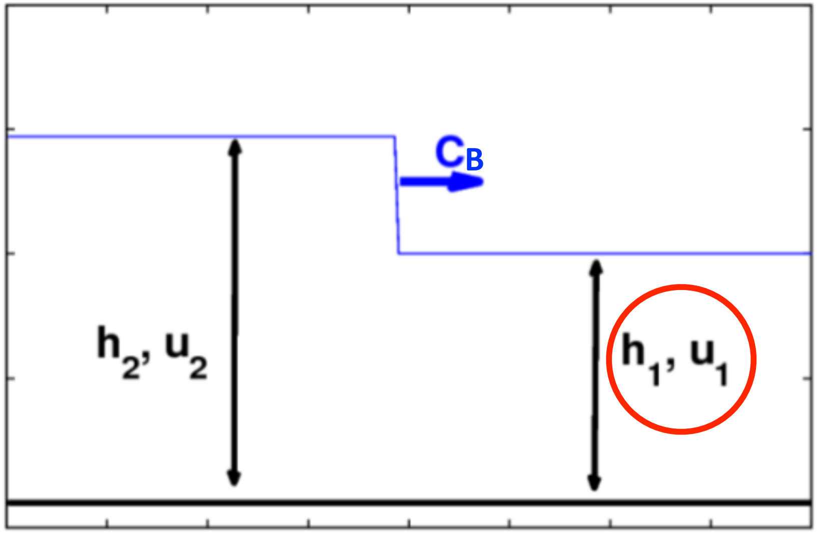
3. discontinuities (bores): Rankine-Hugoniot conditions
4.

Conservation (Mass and momentum)

$$C_B(h_2 - h_1) = (h_2 u_2 - h_1 u_1)$$

$$C_B(h_2 u_2 - h_1 u_1) = (h_2 u_2^2 - h_1 u_1^2 + g h_2^2/2 - g h_1^2/2)$$

Bore: positive surge or hydraulic jump in translation



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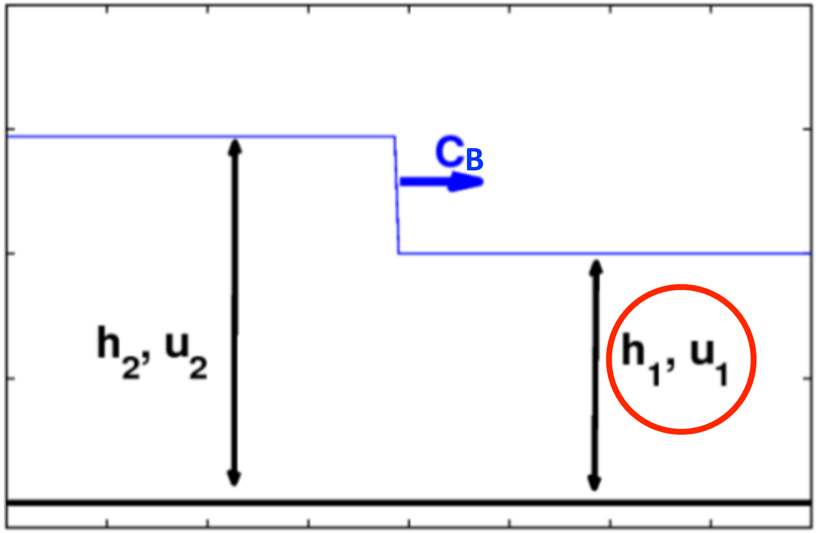

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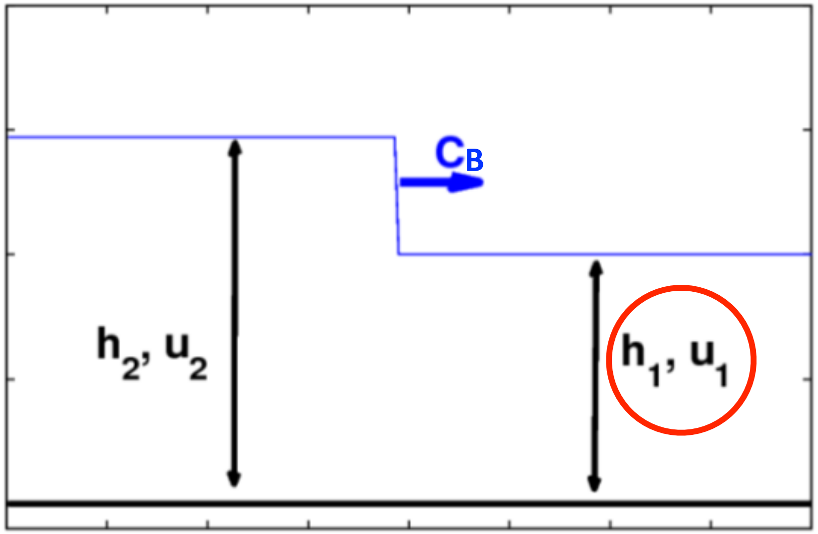
Conservation (Mass and momentum)

$$Fr := \frac{u_1 - C_B}{\sqrt{gh_1}} \quad \Rightarrow \quad C_B \equiv Fr$$

Bore strength

$$\frac{h_2}{h_1} = \frac{\sqrt{1 + 8Fr^2} - 1}{2} \Rightarrow \quad \frac{h_2}{h_1} \equiv Fr$$

Bore: positive surge or hydraulic jump in translation



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4.

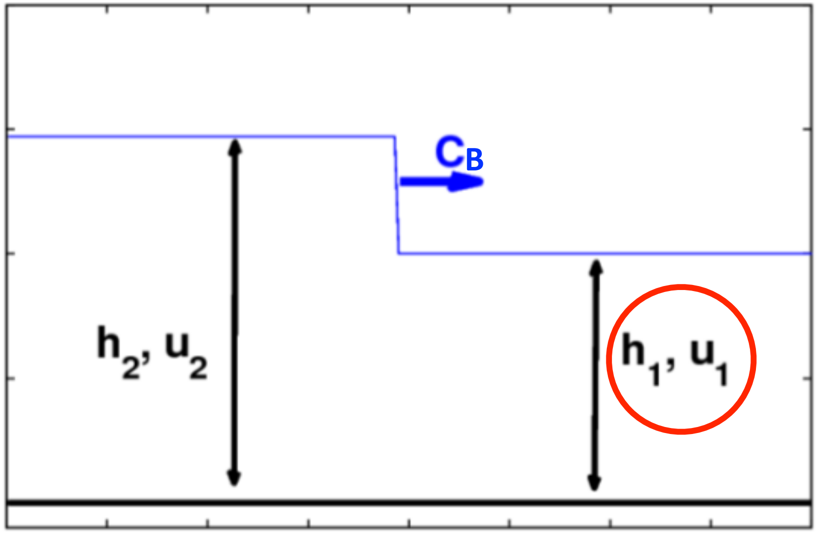
Conservation (Mass and momentum)
and
Dissipation (Energy/entropy)

$$C_B(h_2 - h_1) = (h_2u_2 - h_1u_1)$$

$$C_B(h_2u_2 - h_1u_1) = (h_2u_2^2 - h_1u_1^2 + gh_2^2/2 - gh_1^2/2)$$

$$D_B := C_B(E_2 - E_1) - (F_{E2} - F_{E1}) = -\frac{g}{4}\sqrt{\frac{g\bar{h}}{h_1h_2}}(h_2 - h_1)^3 < 0$$

Bore: positive surge or hydraulic jump in translation



Shallow water equations (hydrostatic, shallow limit)

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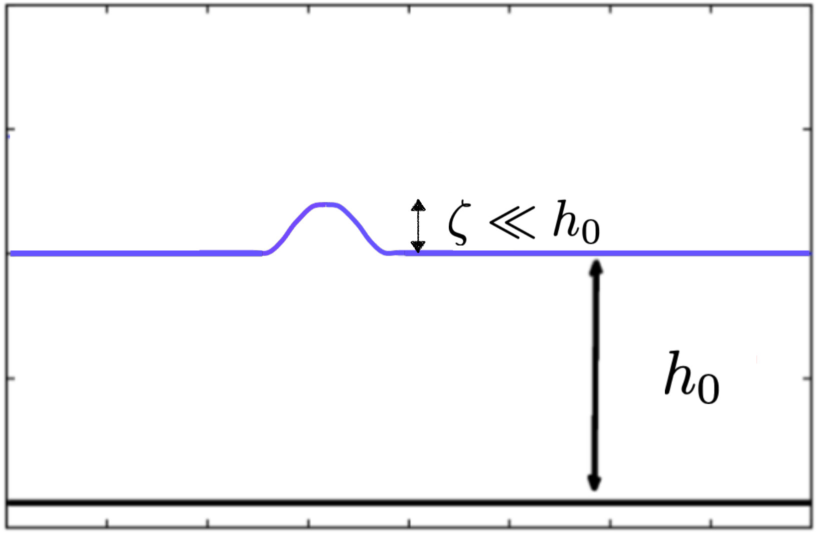

Shallow water equations (hydrostatic, shallow limit)

1. hyperbolic
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4. linear propagation characteristics

Linearized eq.s

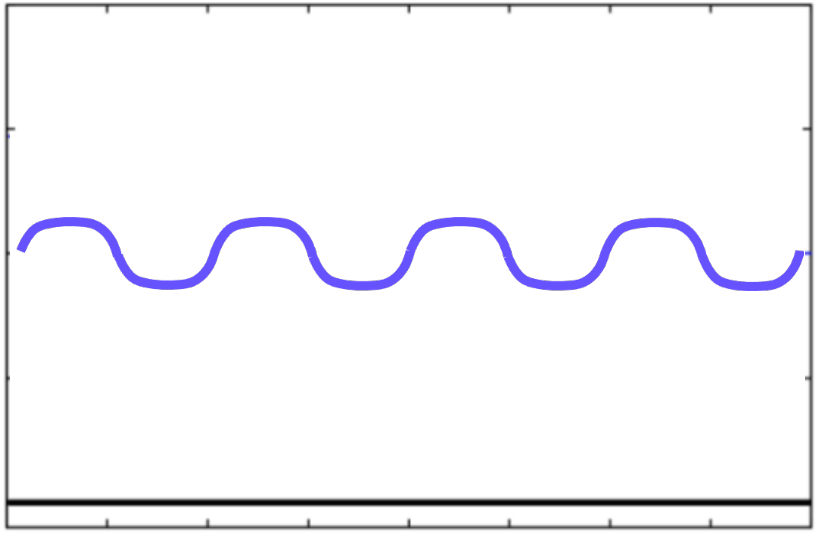
$$h = h_0 + \zeta$$

$$\zeta \ll h_0$$

$$\partial_t \zeta + h_0 \partial_x u = 0$$

$$\partial_t u + g \partial_x \zeta = 0$$

Bore: positive surge or hydraulic jump in translation



Shallow water equations (hydrostatic, shallow limit)

1. hyperbolic
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Linearized eq.s

Fourier mode

$$\zeta = \zeta^* \mathrm{e}^{\mathrm{i}(\kappa x + \nu t)}$$

$$\partial_t \zeta + h_0 \partial_x u = 0$$

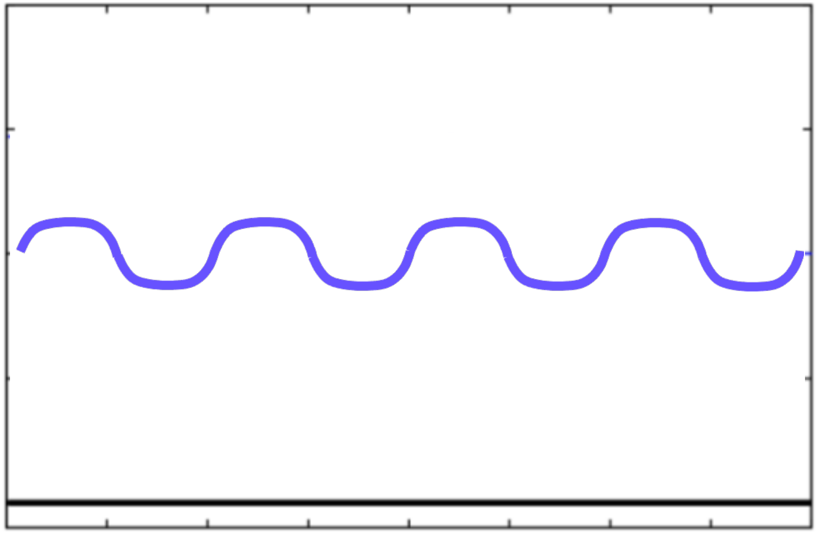
$$u = u^* \mathrm{e}^{\mathrm{i}(\kappa x + \nu t)}$$

$$\partial_t u + g \partial_x \zeta = 0$$

$$\nu = \omega + \mathrm{i}\sigma$$

$$\kappa = \frac{2\pi}{\lambda}$$ wavenumber

Bore: positive surge or hydraulic jump in translation



Shallow water equations (hydrostatic, shallow limit)

1. hyperbolic
2. mathematical entropy (energy)
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Linearized eq.s

Fourier mode

$$\zeta = \zeta^* \mathrm{e}^{\mathrm{i}\kappa(x \pm c_0 t)}$$

$$u = u^* \mathrm{e}^{\mathrm{i}\kappa(x \pm c_0 t)}$$

$$\omega = \pm \kappa c_0 \,, \quad \sigma = 0$$

$$c_0 = \sqrt{g h_0}$$

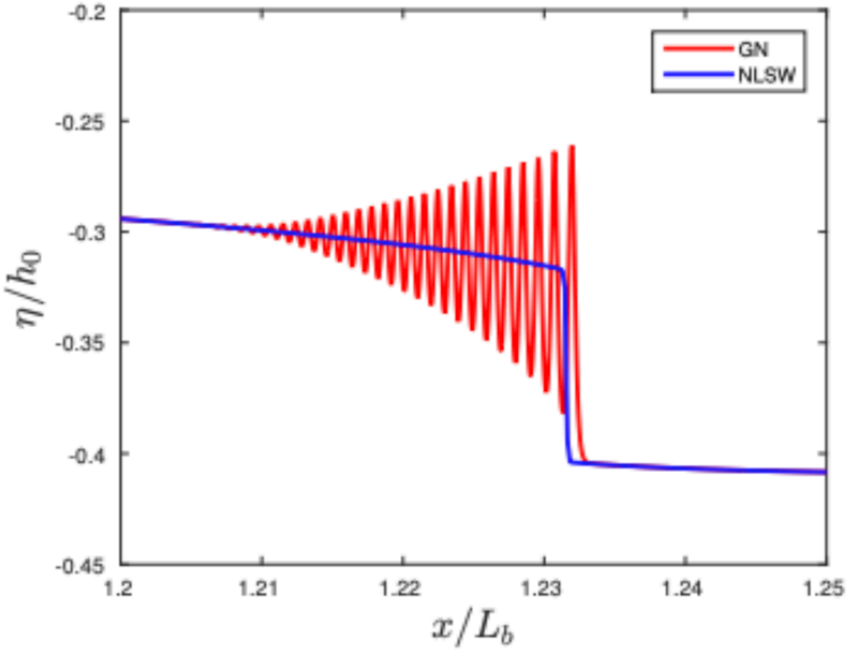
a) No damping

b) Constant celerity c_0

No dispersion …

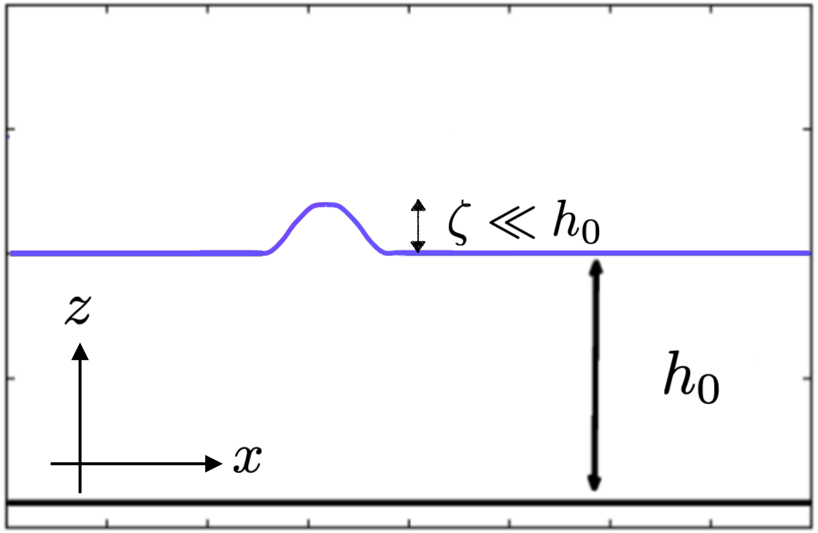
In fluid dynamics, **dispersion** of water **waves** generally refers to frequency **dispersion**, which means that **waves** of different wavelengths travel at different phase speeds. Water **waves**, in this context, are **waves** propagating on the water surface, with gravity and surface tension as the restoring forces.

Dispersion (water waves) - Wikipedia
https://en.wikipedia.org/wiki/**Dispersion**_(water_**waves**)



$$\zeta = \zeta^* e^{i\kappa(x - c(\kappa)t)}$$

Euler equations (Airy theory)



$$\nabla \cdot \vec{\mathsf{v}} = 0$$

$$\partial_t \vec{\mathsf{v}} + (\vec{\mathsf{v}} \cdot \nabla)\vec{\mathsf{v}} + \nabla \tilde{p} = -\, \mathrm{g}\vec{1}_z$$

$$\partial_t \zeta + \mathsf{v}_x \partial_x \zeta = \mathsf{v}_z \qquad z = \zeta$$

$$\tilde{p} = 0 \qquad z = \zeta$$

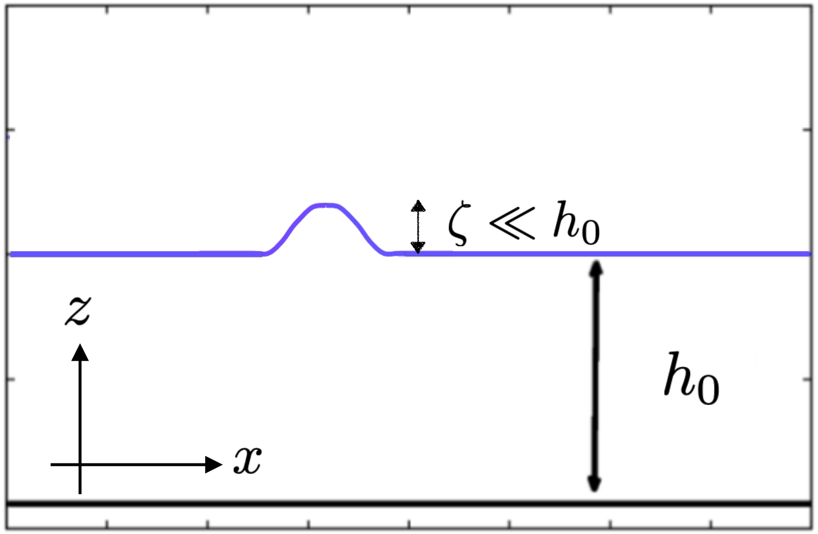
$$\mathsf{v}_z = 0 \qquad z = -h_0$$

$$\zeta = \zeta^* \mathrm{e}^{\mathrm{i}\kappa(x - c(\kappa)t)}$$

Euler equations (Airy theory)

$$\nabla \times \vec{v} = 0 \qquad \vec{v} = \nabla\Phi$$

$$
\begin{aligned}
\Delta\Phi &= 0 \\
\partial_t\Phi + \frac{1}{2}\|\nabla\Phi\|^2 + g\zeta &= 0 \qquad z = \zeta \\
\partial_t\zeta + \partial_x\Phi\partial_x\zeta &= \partial_z\Phi \qquad z = \zeta \\
\partial_z\Phi &= 0 \qquad z = -h_0
\end{aligned}
$$



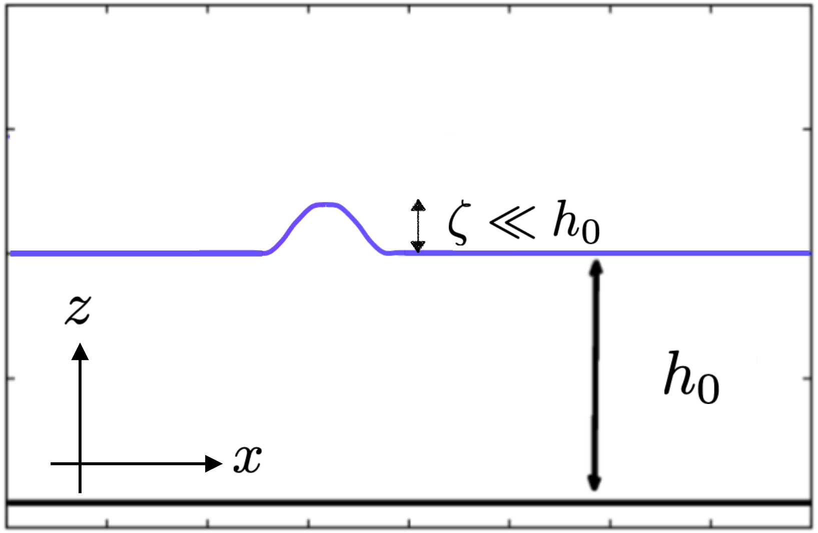
Non-linear potential equations

$$\zeta = \zeta^* e^{i\kappa(x - c(\kappa)t)}$$

Euler equations (Airy theory)

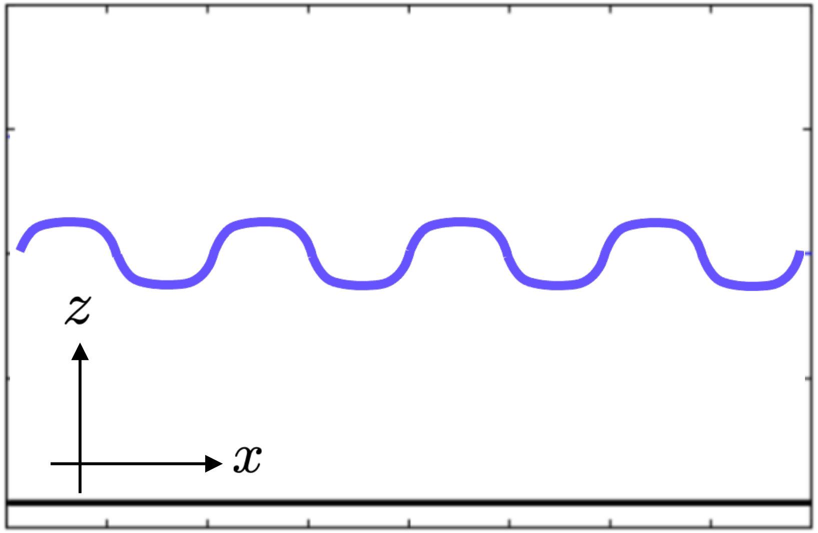
$$\frac{\zeta}{\text{etc}} \quad \frac{\|\nabla\zeta\|}{\|\nabla\Phi\|} \ll 1$$

$$\begin{aligned}
\Delta\Phi &= 0 \\
\partial_t\Phi + g\zeta &= 0 \qquad z = \zeta \\
\partial_t\zeta &= \partial_z\Phi \qquad z = \zeta \\
\partial_z\Phi &= 0 \qquad z = -h_0
\end{aligned}$$



Linear potential equations

$$\zeta = \zeta^* \mathrm{e}^{\mathrm{i}\kappa(x - c(\kappa)t)}$$



Euler equations (Airy theory)

$$\zeta,\quad \|\nabla\zeta\|,\quad \|\nabla\Phi\|,\quad \text{etc} \ll 1$$

$$\begin{aligned}
\Delta\Phi &= 0 \\
\partial_t\Phi + g\zeta &= 0 \qquad z = \zeta \\
\partial_t\zeta &= \partial_z\Phi \qquad z = \zeta \\
\partial_z\Phi &= 0 \qquad z = -h_0
\end{aligned}$$

Linear potential equations

$$\Phi = B(z)\Phi^* e^{i(kx+\nu t)}$$

$$\zeta = \zeta^* e^{i(kx+\nu t)}$$

$$\zeta^* \ll 1$$

$$c^2(\kappa) = g h_0 \frac{\tanh(\kappa h_0)}{\kappa h_0}$$

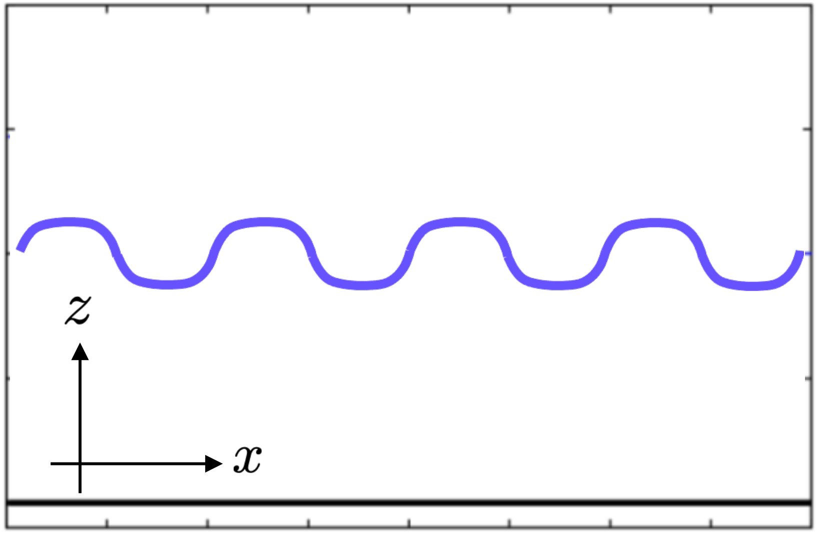
$$\zeta = \zeta^* \mathrm{e}^{\mathrm{i}\kappa(x - c(\kappa)t)}$$

Stoker, John Wiley & Sons, 1992

Euler equations (Airy theory)

$$\zeta,\quad \|\nabla\zeta\|,\quad \|\nabla\Phi\|,\ \text{etc} \ll 1$$

$$\begin{aligned} \Delta\Phi &= 0 & & \\ \partial_t\Phi + g\zeta &= 0 & z &= \zeta \\ \partial_t\zeta &= \partial_z\Phi & z &= \zeta \\ \partial_z\Phi &= 0 & z &= -h_0 \end{aligned}$$



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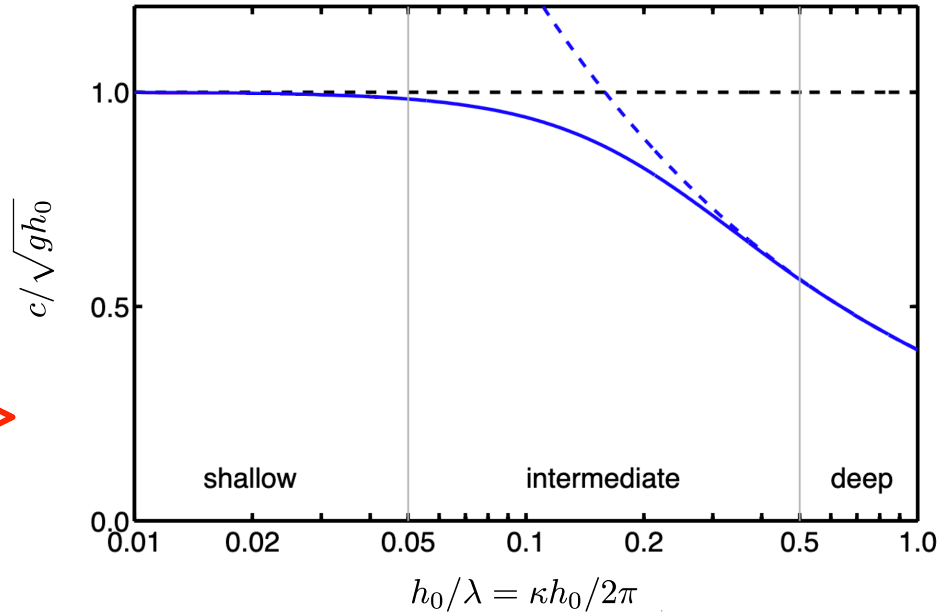

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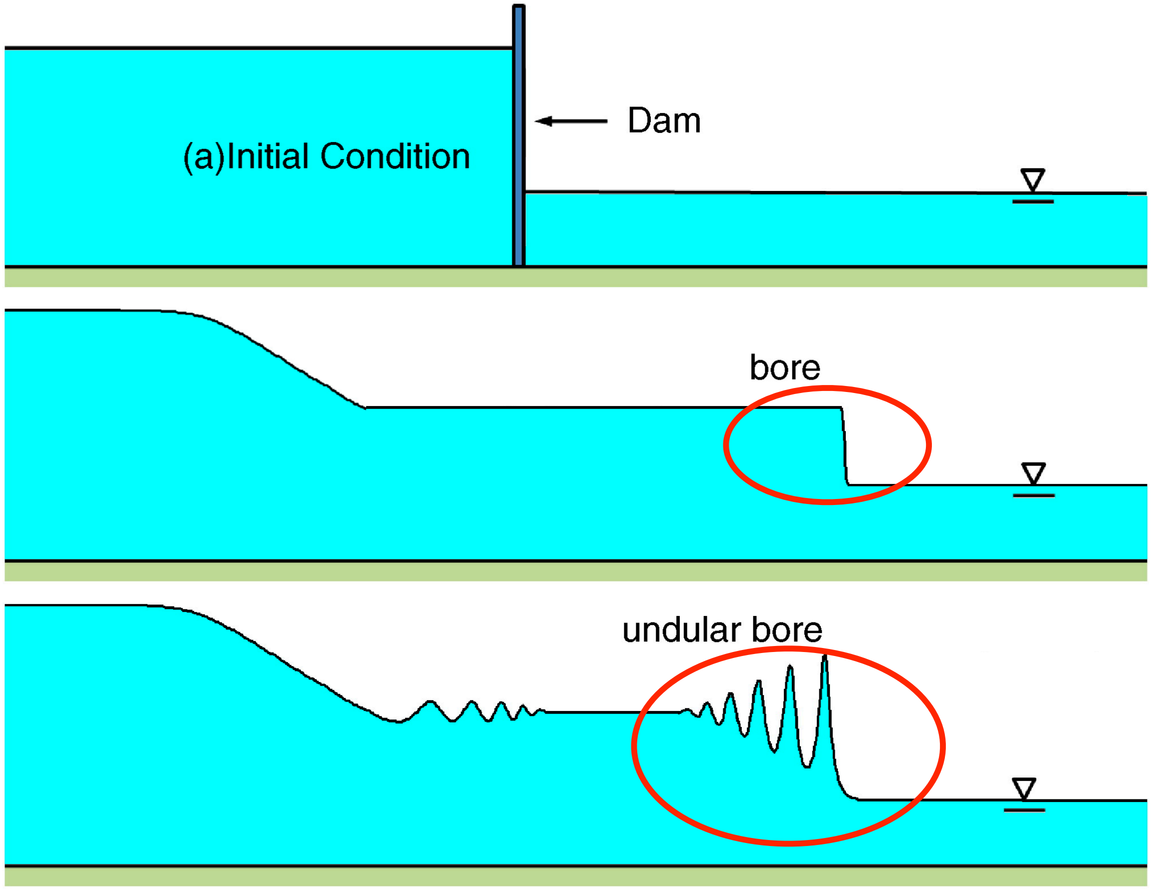
Stoker, John Wiley & Sons, 1992

Bores in rectangular channels (1D/no banks)



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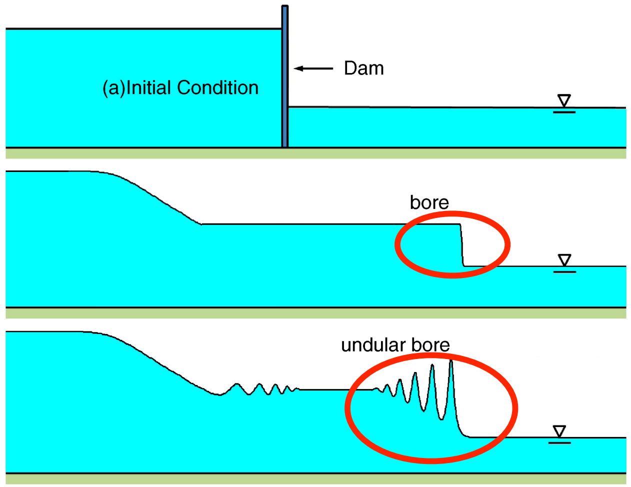


- **Mathematical and physical theories**:

Rayleigh 1914, **Lemoine** 1948, **Benjamin & Lighthill** 1954, **Serre** 1954, **Johnson** 1970, **Gurevich & Pitaevskii** 1973, **El et al.** 2006, **Congy et al** 2021,
and many others

- **Laboratory experiments**:

Favre 1935, **Sandover and Zienkiewics** 1957, **Bennet & Cunge** 1971, **Treske** 1994, **Chanson** 1996 & 2009, **Soares Frazao and Zech** 2002, **Simon** 2013, **Furgerot** 2014, **David et al.** 2014,
and many others …

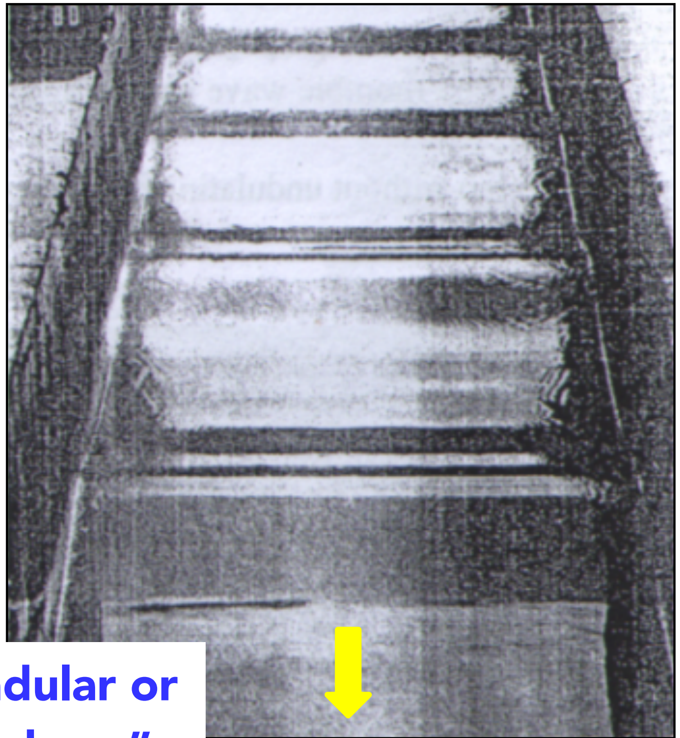
- **Numerical simulations**:

Peregrine 1966, **Wei et al.** 1995, **Soares Frazao and Zech** 2002, **Lubin et al.** 2010, **Pan & Lu** 2011, **Tissier et al.** 2011, **Simon** 2013, **Filippini et al.** 2019,
and many others …

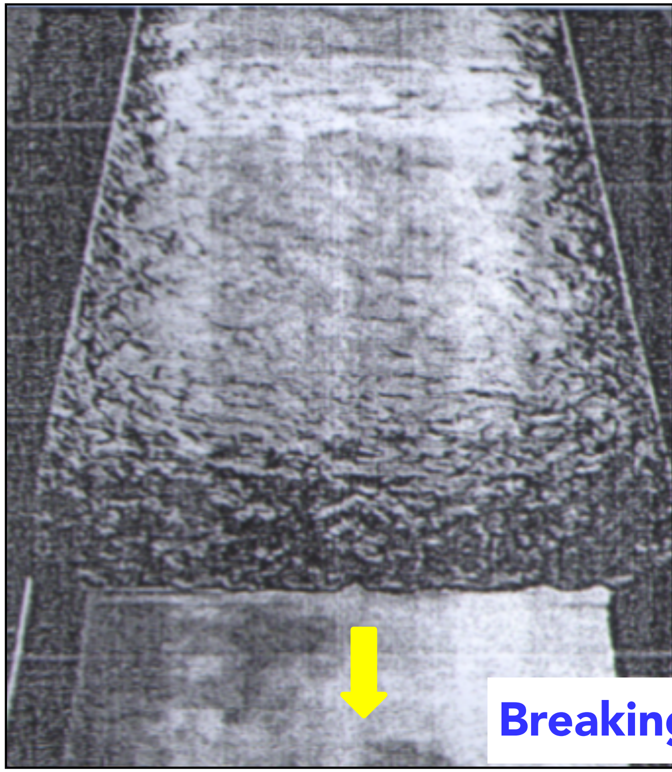
Experiments in rectangular channels (no banks)

Favre, *Dunod, 1935*

Treske, *J. Hydraulic Research, 1994*



classical undular or "dispersive bore" or "Favre wave"



Breaking bore

$F_{t2} > Fr > 1$ — F_{t2} — $Fr > F_{t2}$ → Fr

Nonlinearity vs dispersion | Experimental studies

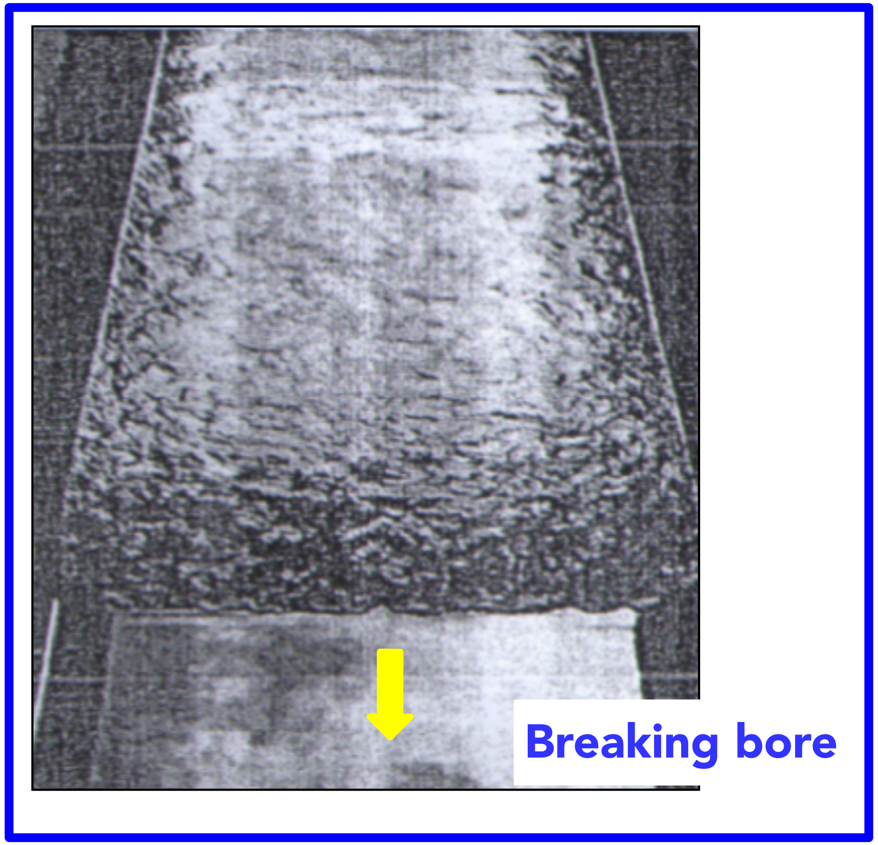
annels (no banks)

Qiantang River - China

Kampar River - Sumatra

Favre, *Dunod, 1935*

Treske, *J. Hydraulic Research, 1994*



F_{t2}

$Fr > F_{t2}$

Fr

Experiments in rectangular channels (no banks)

Favre, *Dunod, 1935*

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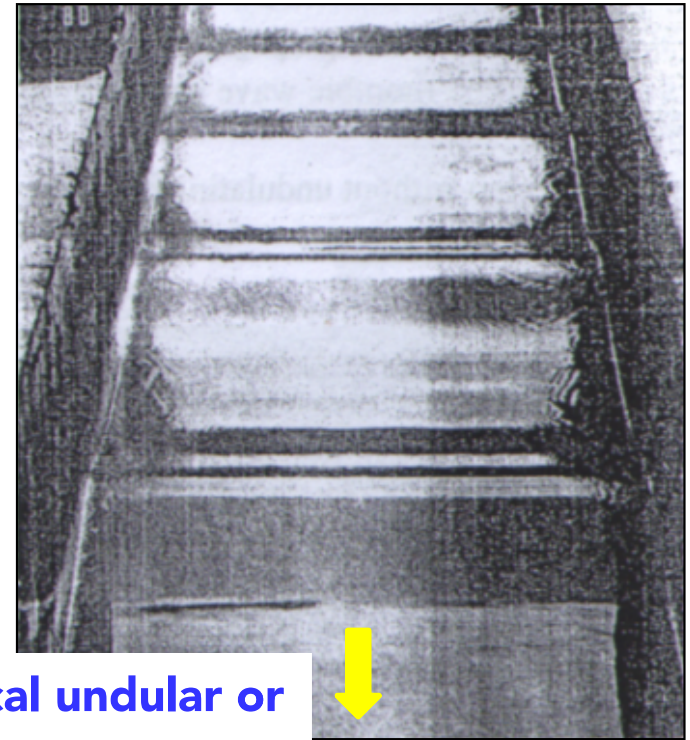

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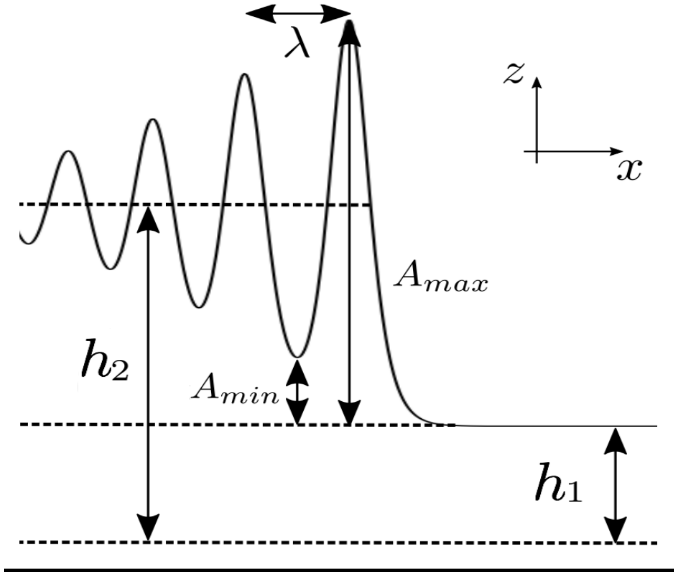

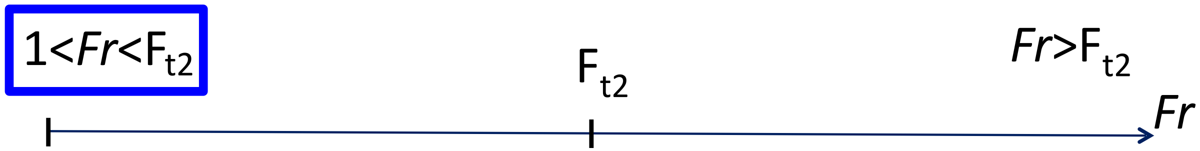

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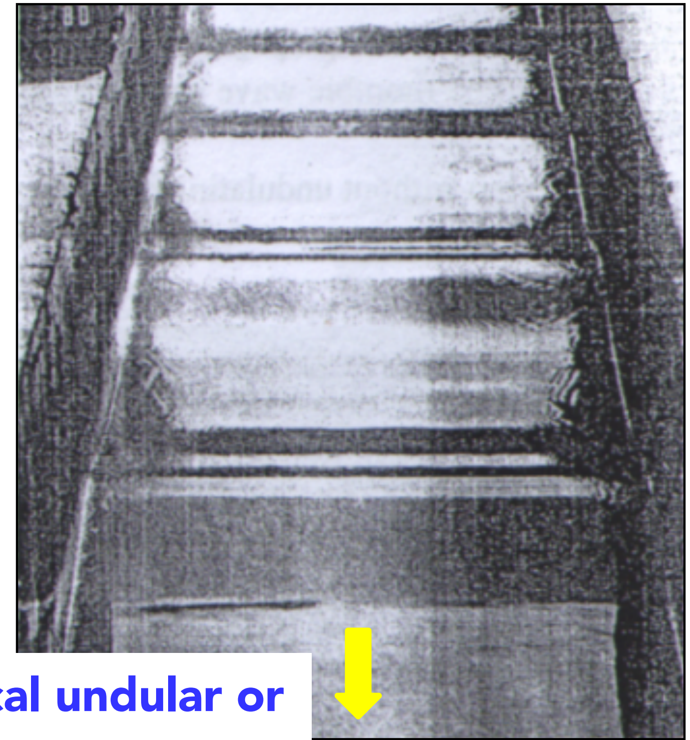

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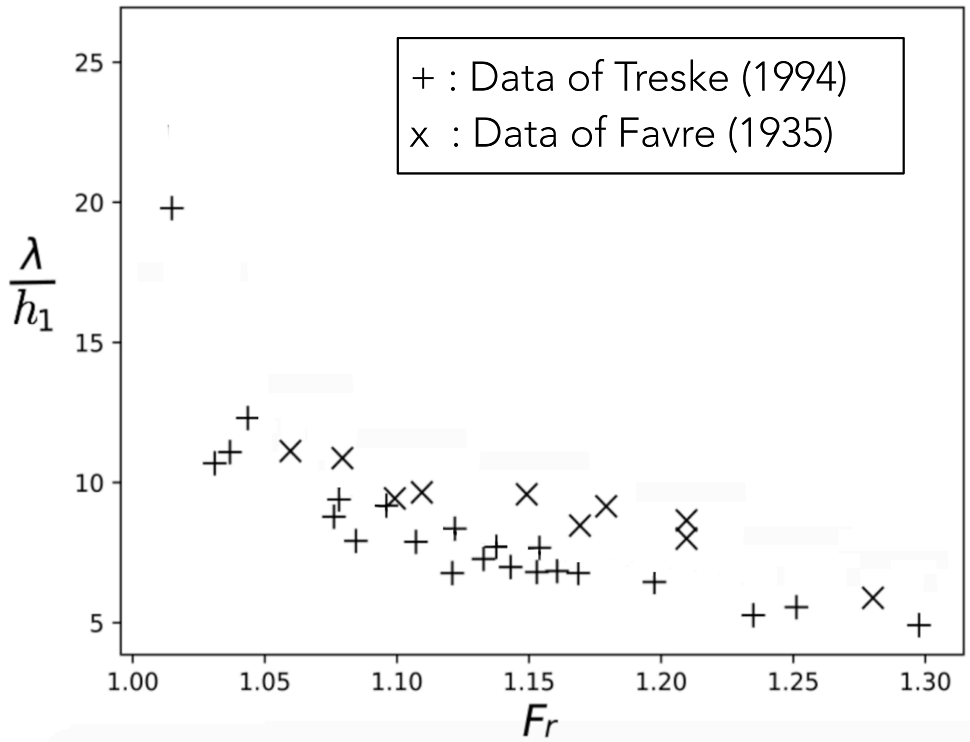

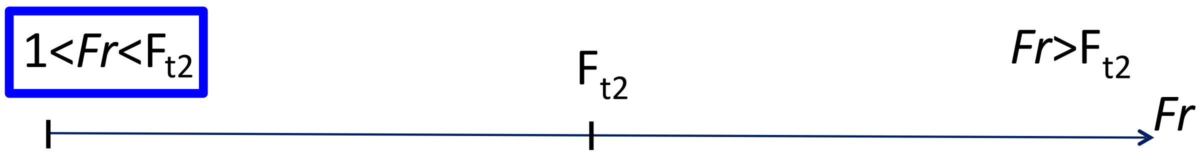

Some physical insight: Lemoine analogy

Lemoine, La Houille Blanche, 1948

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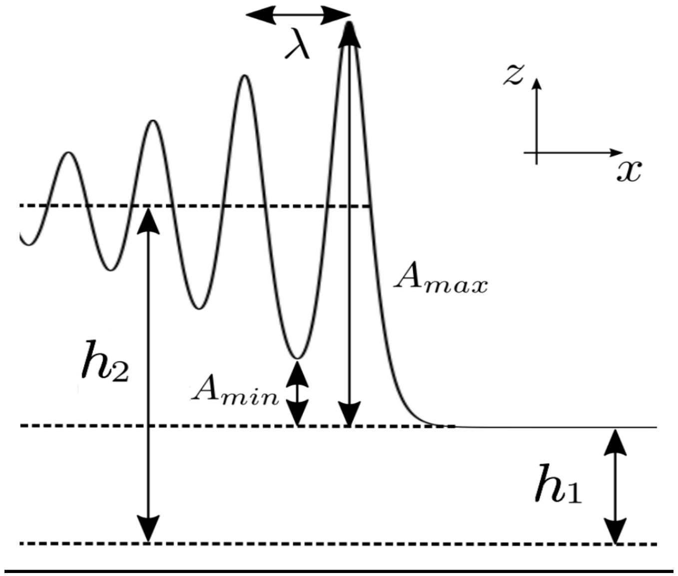
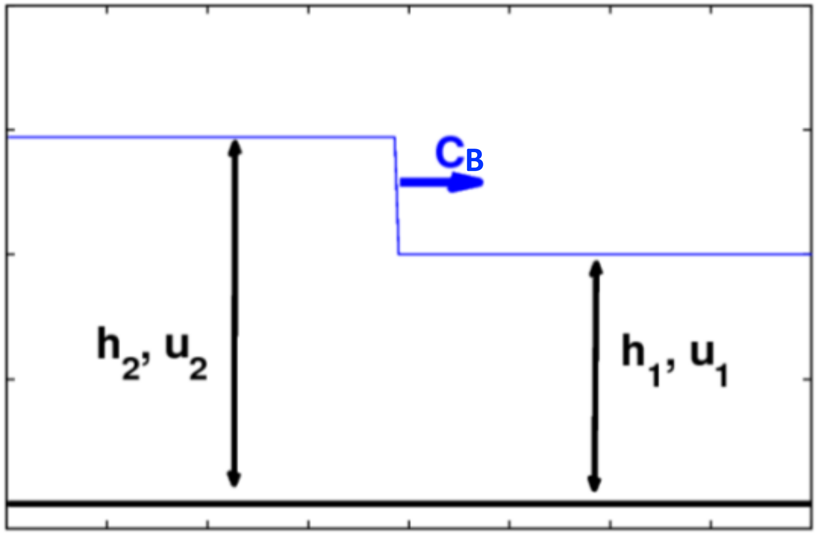
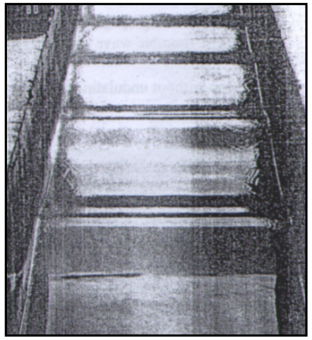
Lemoine, La Houille Blanche, 1948

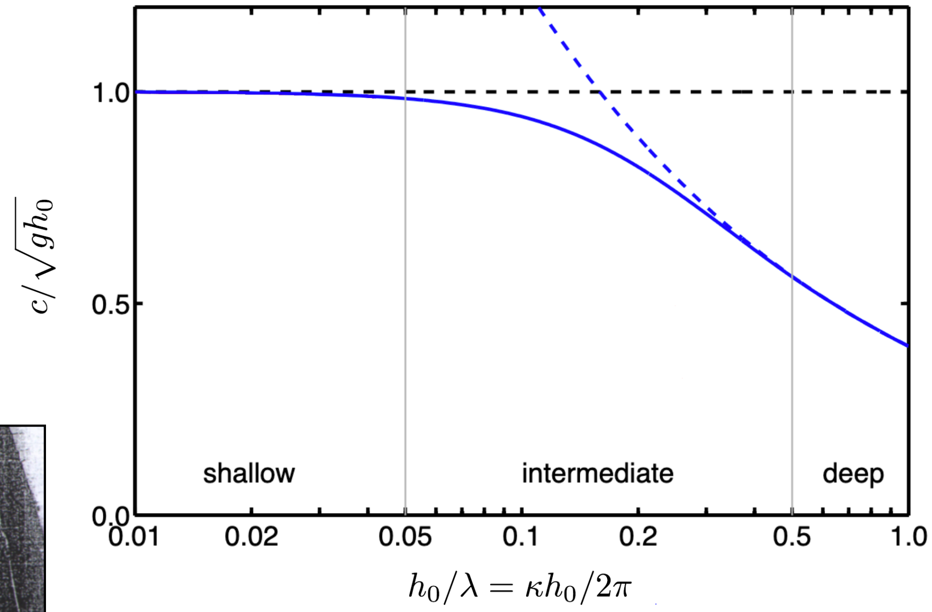


1. Secondary waves conserve mass/momentum
2. The undular front moves at the speed corresponding to conservation
3. The energy normally dissipated goes into the secondary waves

Lemoine analogy







Bore:
given the upstream/downstream conditions and using the jump conditions:

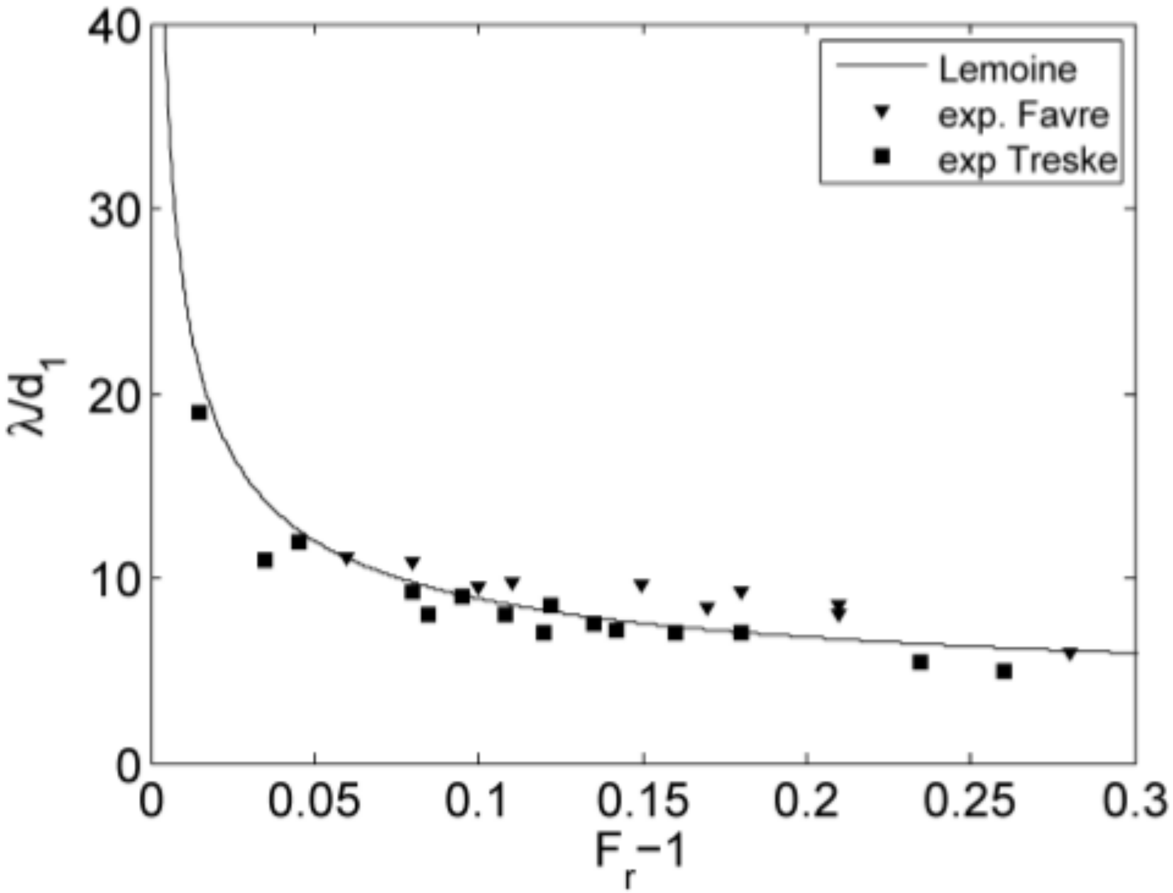
$$C_B = C_B(Fr) = C_B(h_2/h_1)$$

Water waves:
dispersion relation based on the linearized Euler equations (Airy theory)

$$C = C(\lambda) = \sqrt{\frac{g\lambda}{2\pi}\tanh(2\pi h/\lambda)}$$

$$C_B(Fr) = C(\lambda) \Longrightarrow \lambda(Fr)$$

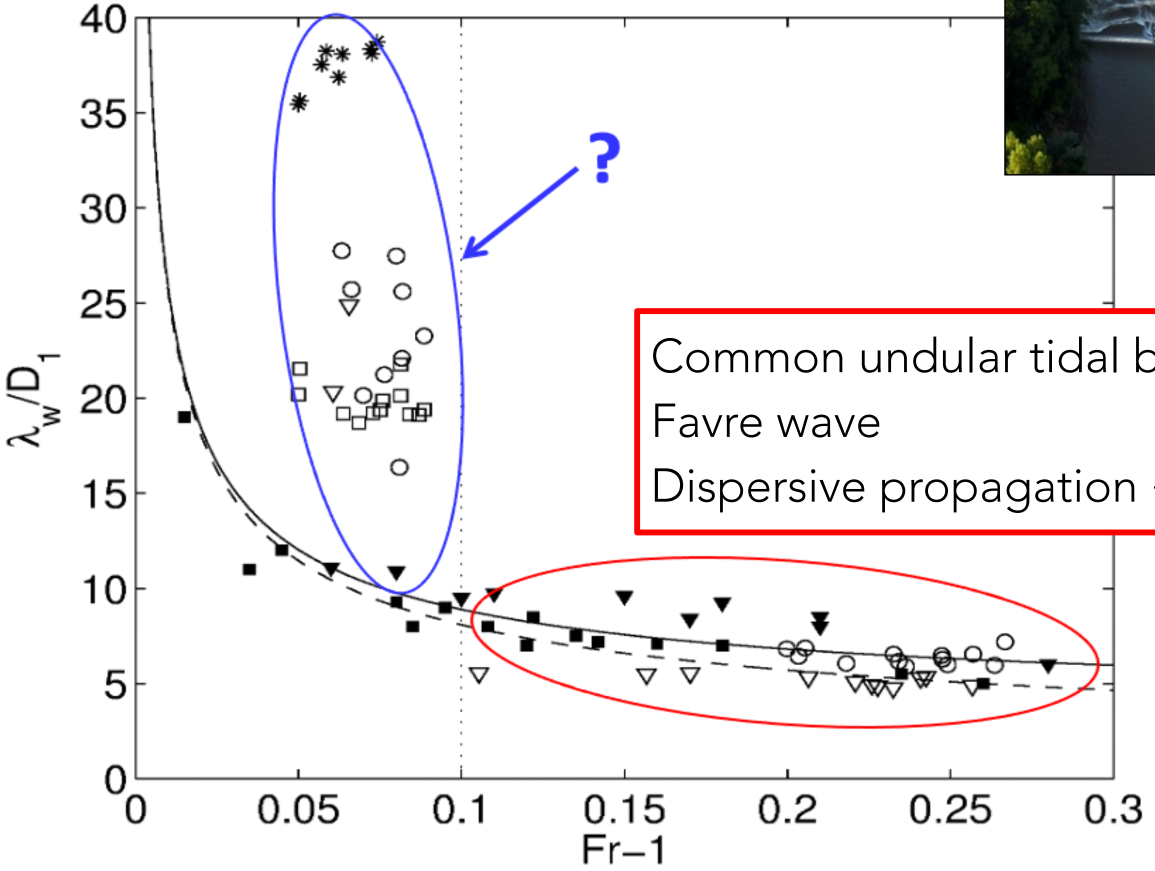
Lemoine, La Houille Blanche, 1948


40
30
20
10
0
0
0.05
0.1
0.15
0.2
0.25
0.3
λ/d_1
F_r-1
Lemoine
exp. Favre
exp Treske

$$\lambda(Fr)$$

Bonneton et al, *Comptes Rendus Geoscience*, 2012

Bonneton et al, *J. Geophysical Research - Oceans*, 2015



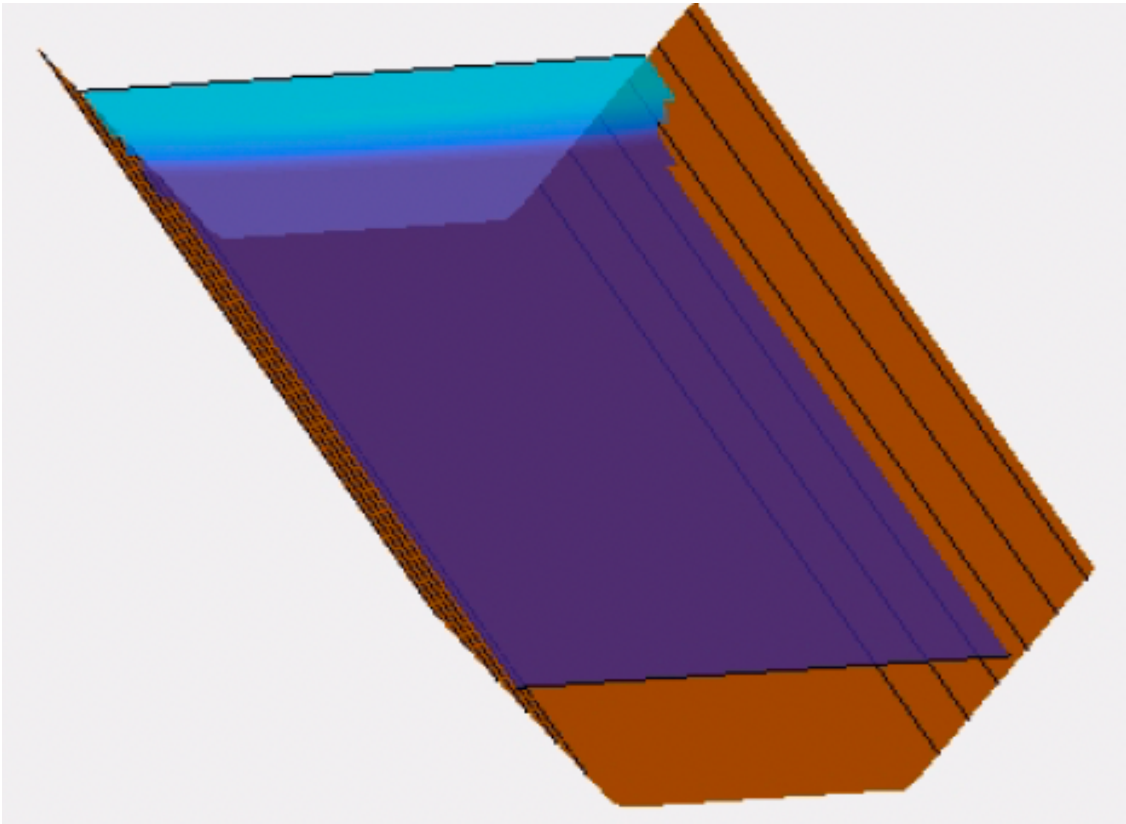
Common undular tidal bore (mascaret):
Favre wave
Dispersive propagation + nonlinearity

$$\lambda(Fr)$$

Favre experiments and low Froude transition

Experiments in trapezoidal channels

Treske, *J. Hydraulic Research, 1994*



Experiments in trapezoidal channels

Treske, *J. Hydraulic Research, 1994*

Fig. 8. Undular bore at Froude ~ 1.04.

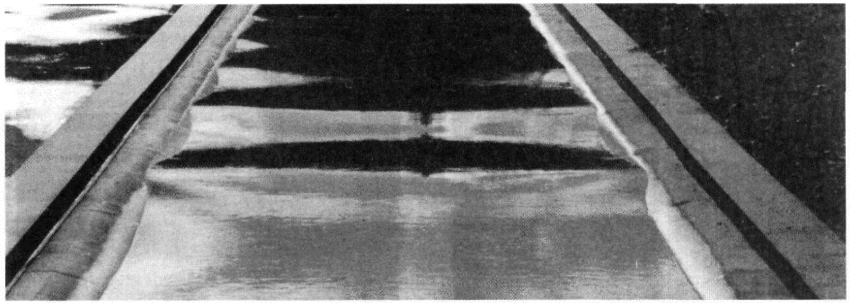


Fig. 9. Undular bore at Froude ~ 1.06.

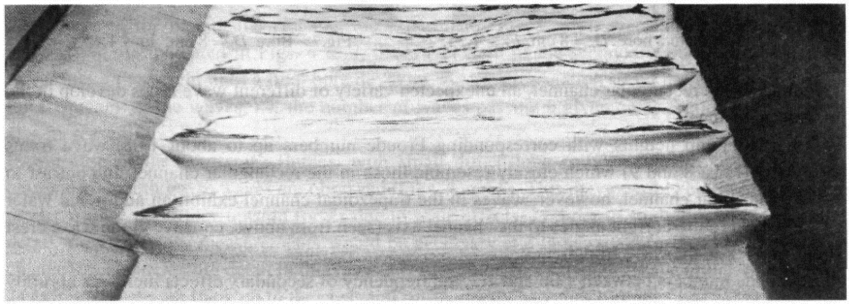


Fig. 10. Undular bore at Froude ~ 1.10.

Fr

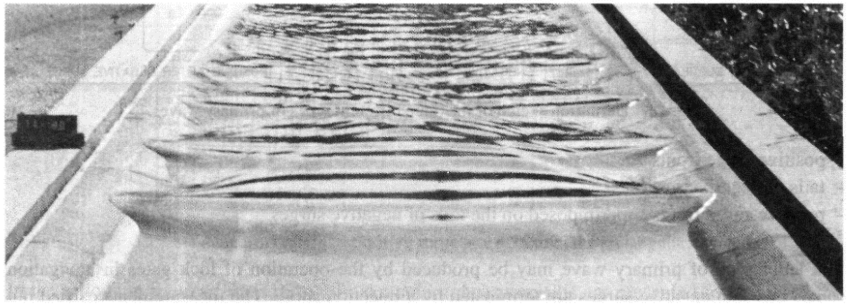


Fig. 11. Undular bore at Froude ~ 1.12.

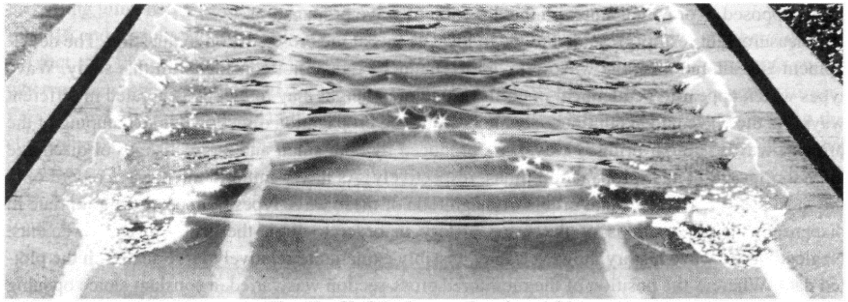


Fig. 12. Undular bore at Froude ~ 1.24.

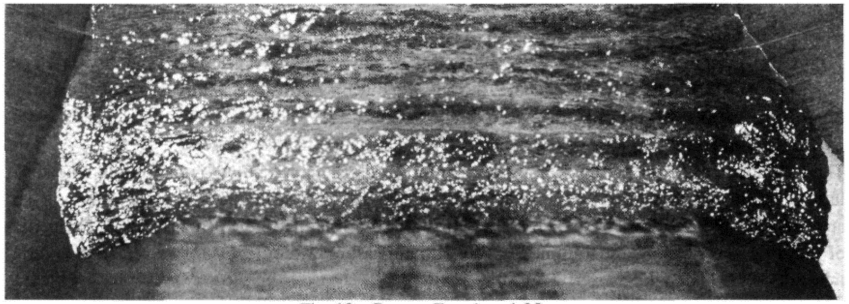


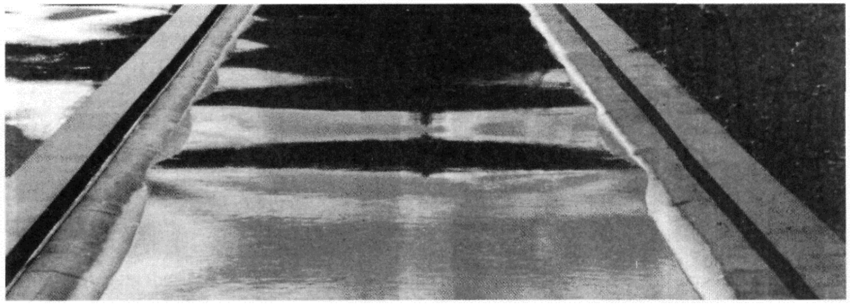
Fig. 13. Bore at Froude ~ 1.35.

$1 < Fr < F_{t1}$ | F_{t1} | $F_{t1} < Fr < F_{t2}$ | F_{t2} | $Fr > F_{t2}$ | Fr

Experiments in trapezoidal channels

Treske, *J. Hydraulic Research, 1994*


Fig. 8. Undular bore at Froude ~ 1.04.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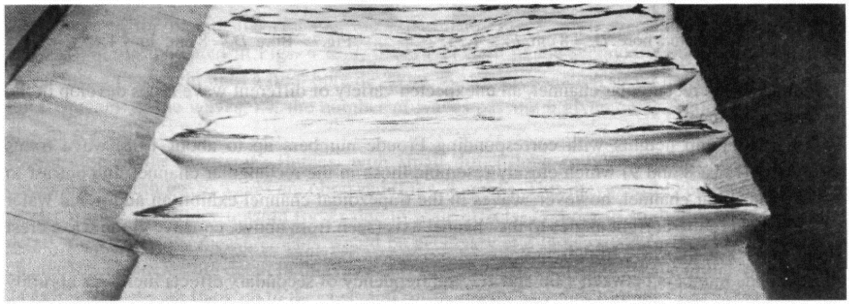
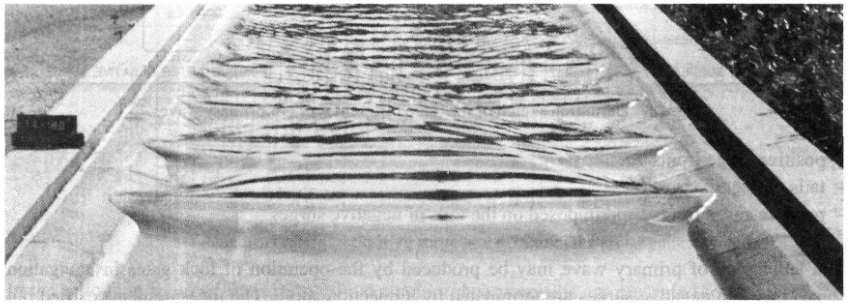


Fig. 10. Undular bore at Froude ~ 1.10.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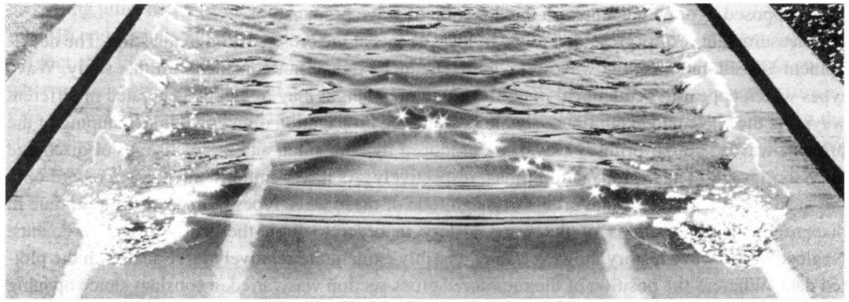
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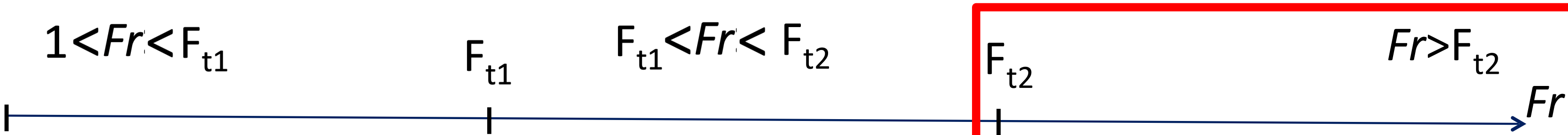

Experiments in trapezoidal channels

Treske, *J. Hydraulic Research, 1994*

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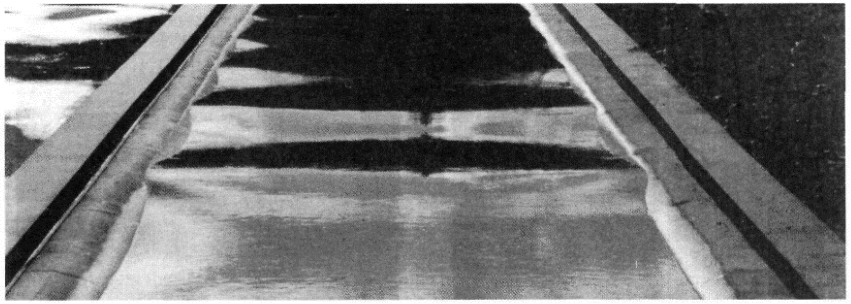

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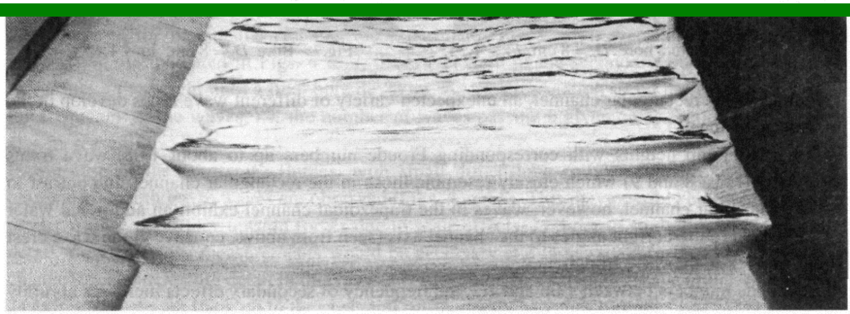

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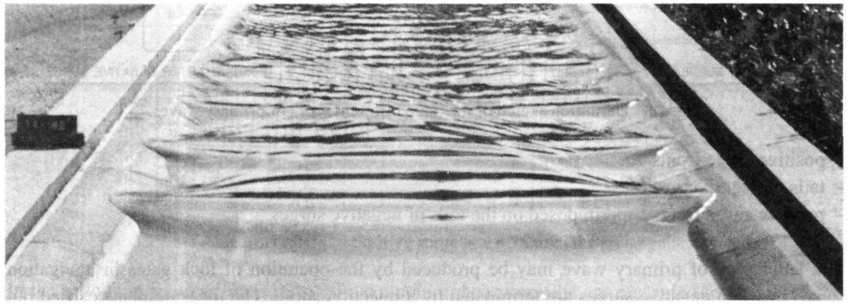

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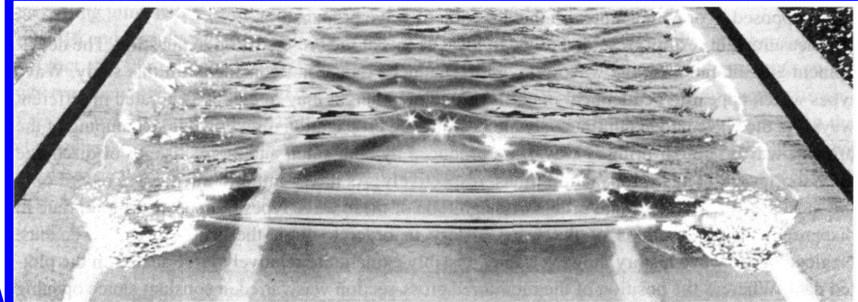

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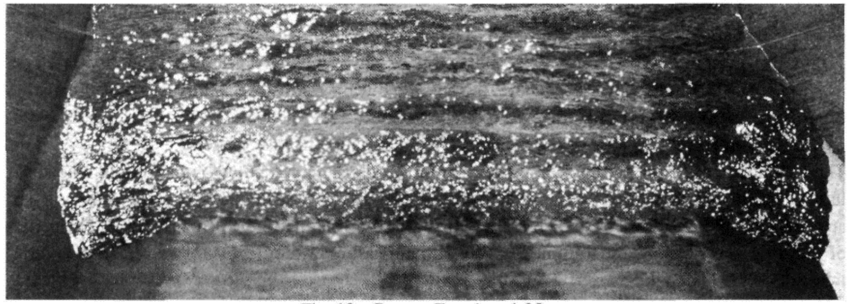

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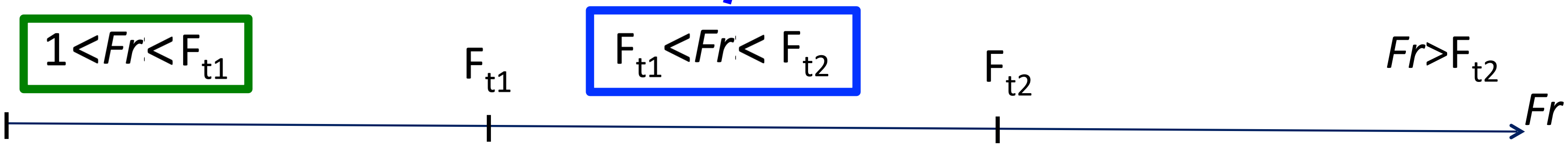
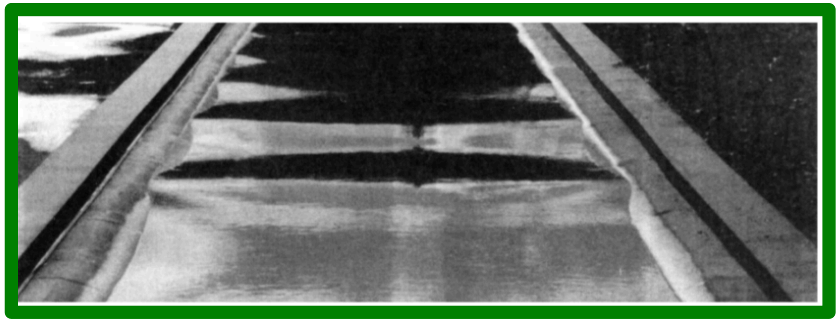
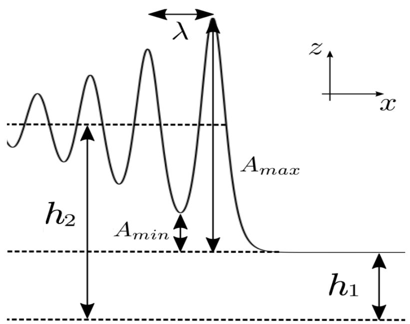
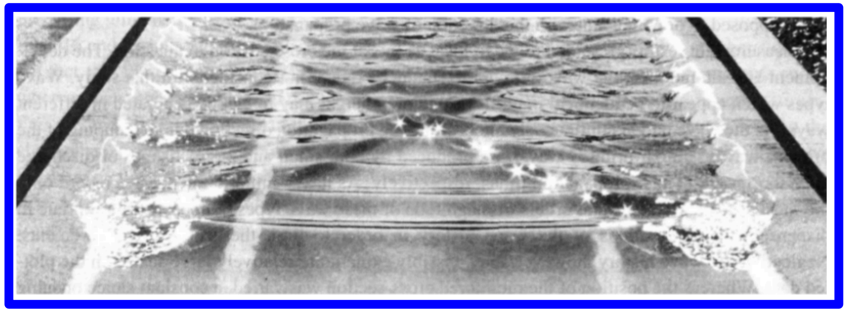


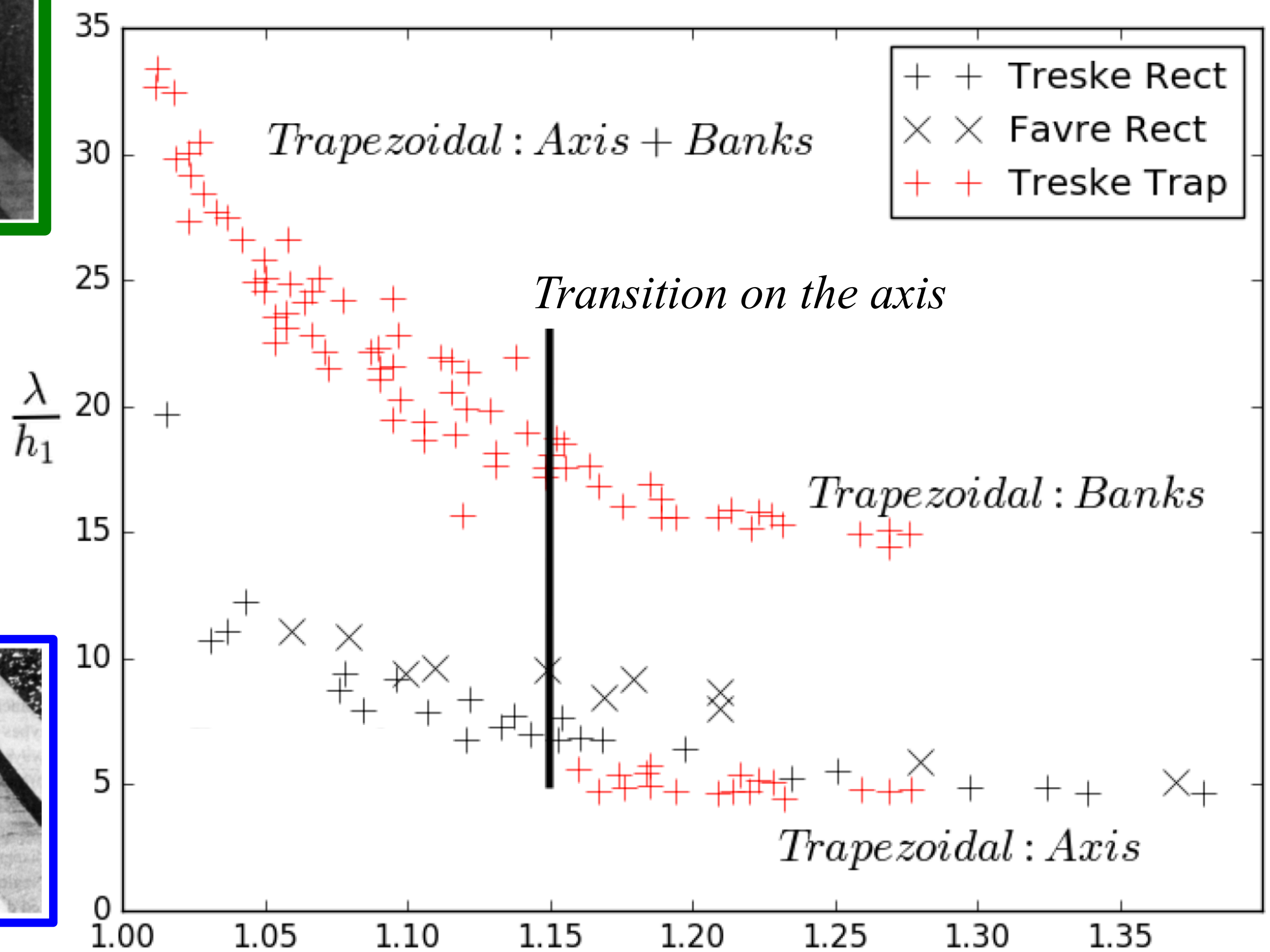
Experiments in trapezoidal channels

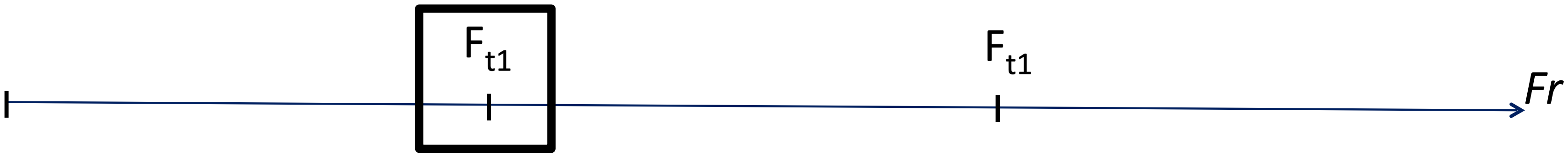
Treske, *J. Hydraulic Research, 1994*



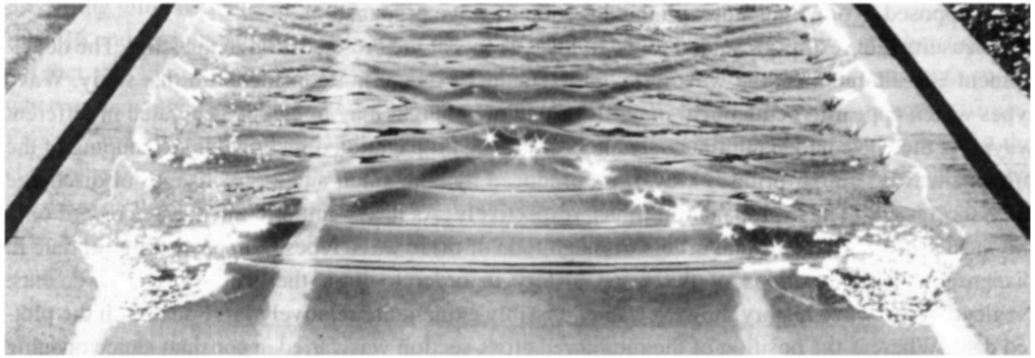


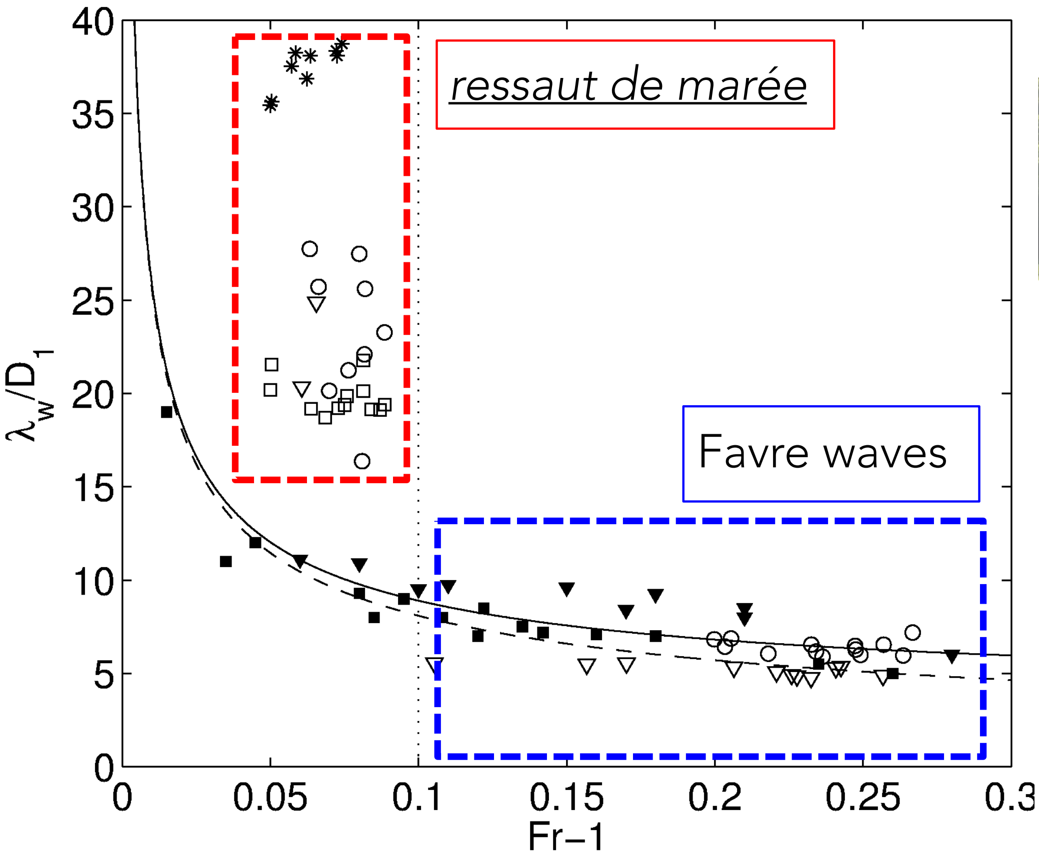




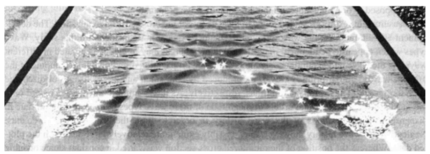


Striking similarities between the low Fr transition observed in field and laboratory experiments

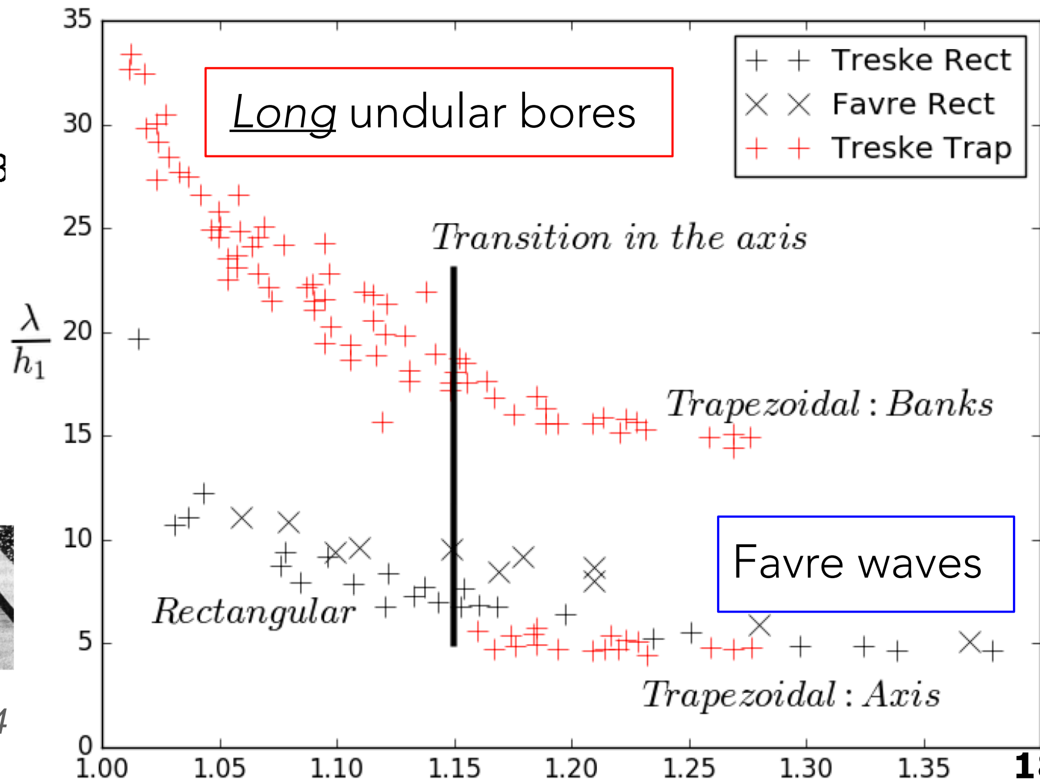




Bonneton et al, *J. Geophysical Research - Oceans*, 2015

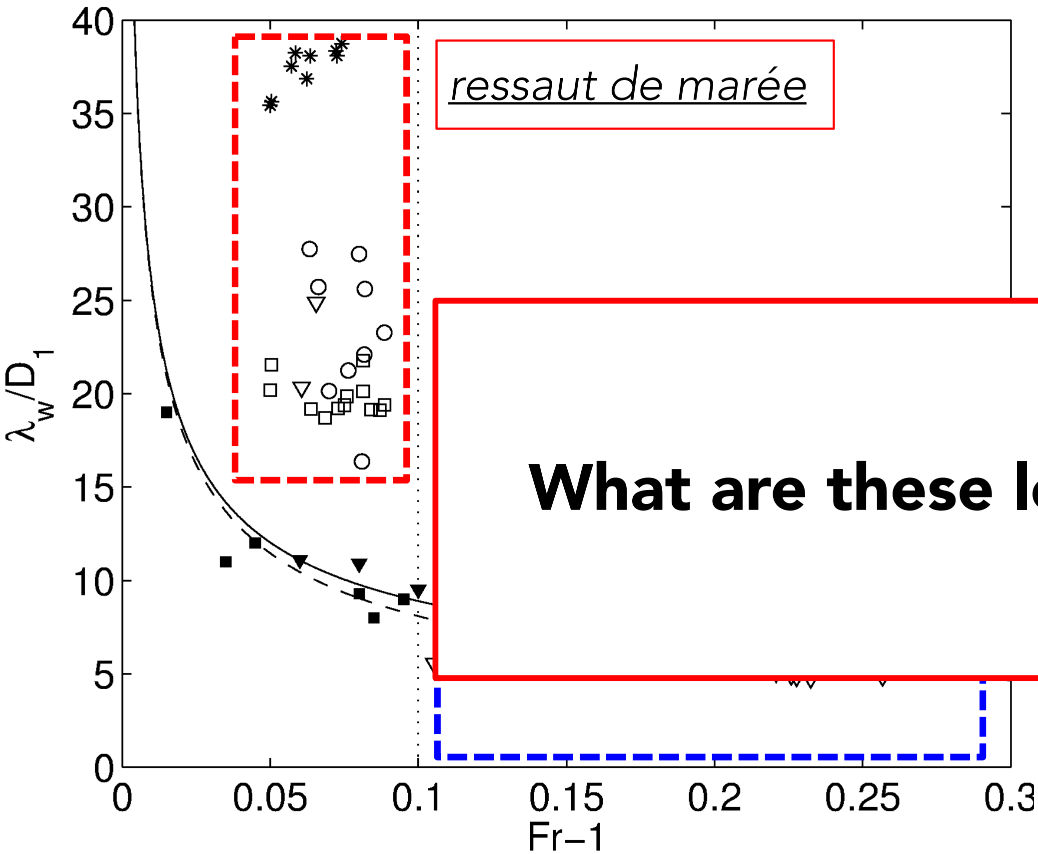


Treske, *J. Hydraulic Research, 1994*



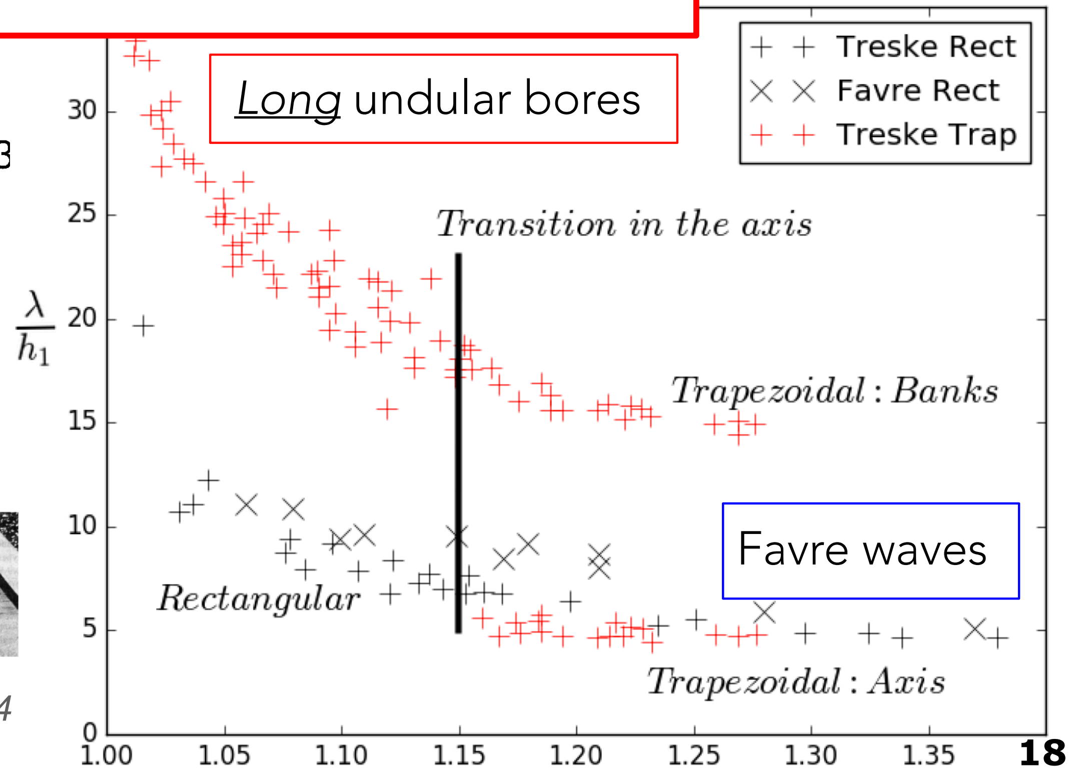
Low Fr Transition

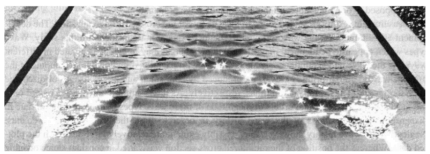
Experimental studies



Bonneton et al, *J. Geophysical Research - Oceans*, 2015

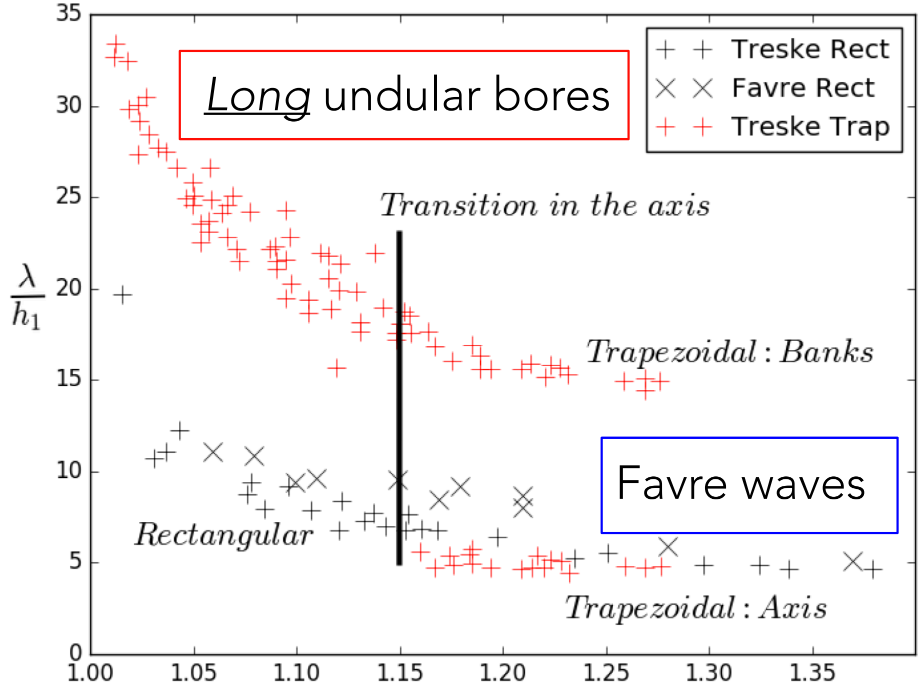
What are these low Froude waves ?





Treske, *J. Hydraulic Research, 1994*

Dispersive wave models



- Complex free surface dynamics (2D)
- Variable flow parameters: *Fr*
- Variable geometrical parameters: channel geom.

Using full 3D models: overkill

→ **Approximate 2d models**

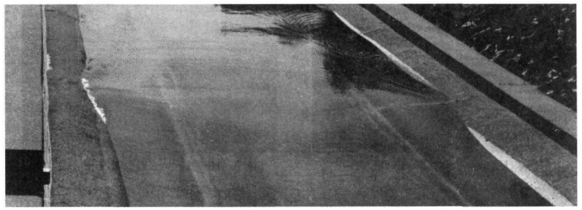
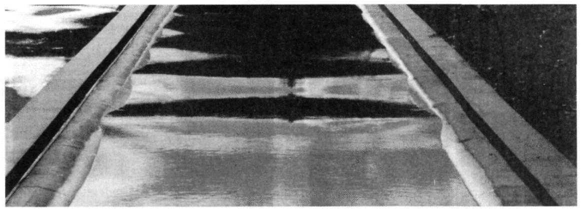


Fig. 8. Undular bore at Froude ~ 1.04.



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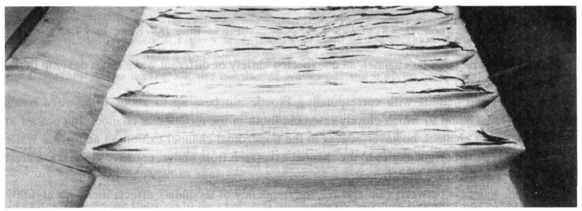

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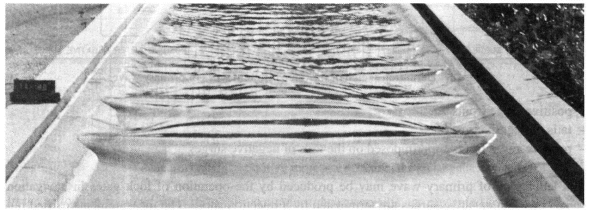

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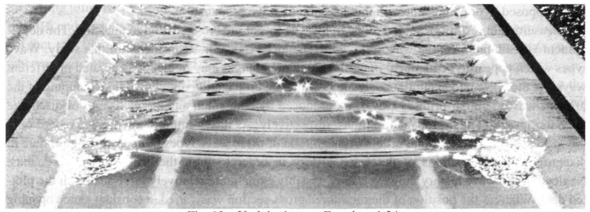

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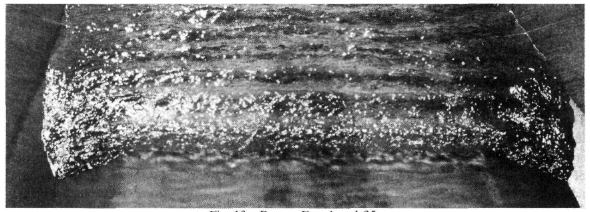

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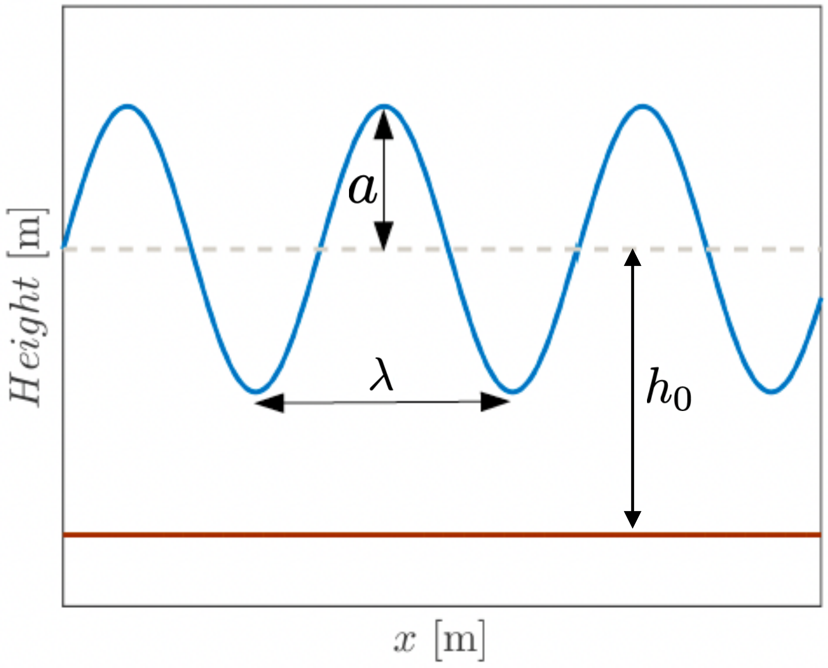


Dimensionless parameters

- dispersion: $\mu = \dfrac{h_0}{\lambda} = \dfrac{\kappa h_0}{2\pi}$
- non-linearity: $\epsilon = \dfrac{a}{h_0}$

Physical hypotheses

Long waves : small μ

Weakly dispersive waves : $\mu^2 \ll 1$, μ^4 negligible

Weak/full non-linearity : $\epsilon = \mathcal{O}(\mu^2)$ and $\epsilon = \mathcal{O}(1)$ respectively

Principles: asymptotic expansion, depth averaging

Boussinesq, J.Math. Pures Appl., 1872
Dingemans, World Scientific, 1997
Lannes, AMS, 2013
Lannes, Nonlinearity, 2020

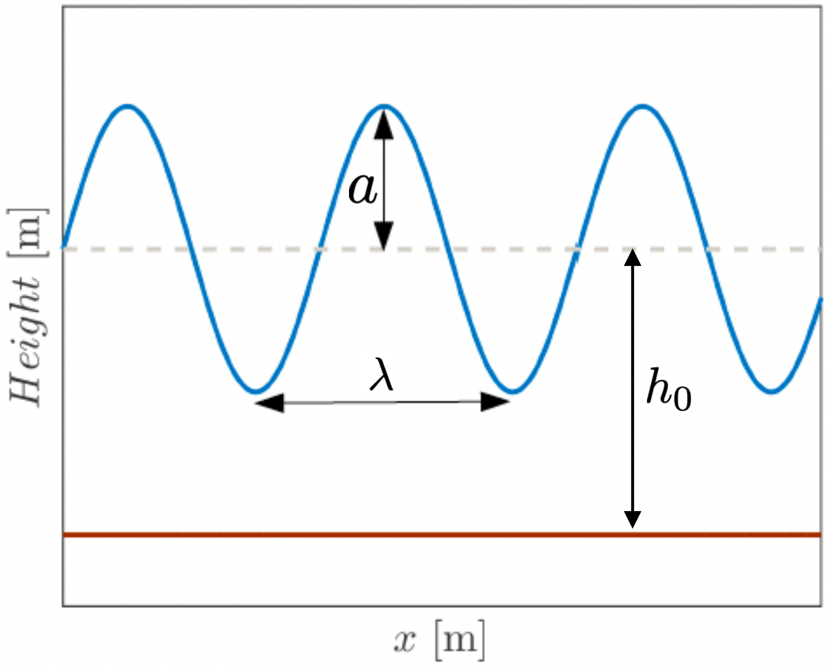
1. Starting point : nonlinear wave equations

$$\begin{aligned} \Delta\Phi &= 0 \\ \partial_t\Phi + \frac{1}{2}\|\nabla\Phi\|^2 + g\zeta &= 0 \\ \partial_t\zeta + \partial_x\Phi\partial_x\zeta &= \partial_z\Phi \\ \partial_z\Phi &= 0 \end{aligned}$$

2. Asymptotic dev. wrt μ^2 : $\Phi = \Phi_0 + \mu^2\Phi_1 + \mu^4\Phi_2 + \ldots$

3. Depth averaging : $\int_0^{h_0+\zeta} (\cdot)\, dz$ → $h\vec{u} = \int_b^{\zeta} \vec{v}\, dz$

4. Retain appropriate order terms



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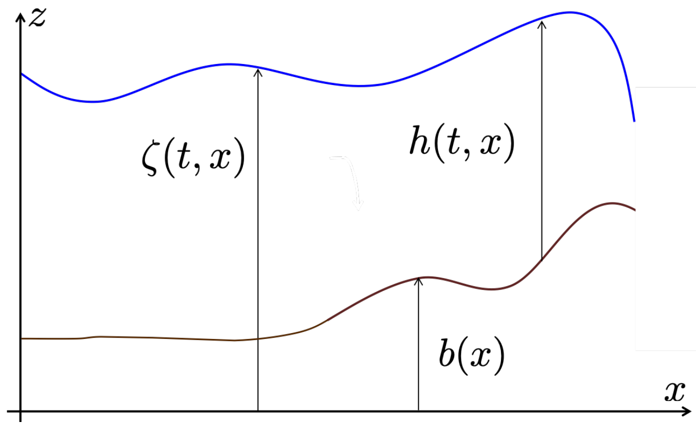


Zeroth order in μ

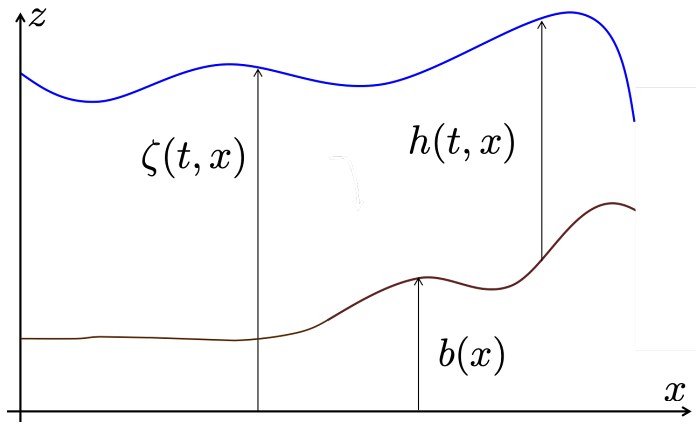
Shallow water/Saint Venant equations

$$\partial_t h + \partial_x (h\mathsf{u}) = 0$$

$$\partial_t (h\mathsf{u}) + \partial_x (h\mathsf{u}^2 + gh^2/2) + gh\partial_x b = 0$$

Bathymetry

Depth averaged velocity $h\vec{\mathsf{u}} = \int_b^\zeta \vec{\mathsf{v}}\, dz$



Weakly dispersive (μ^2) corrections

Weakly nonlinear: $\epsilon = \mathcal{O}(\mu^2)$

$$\partial_t h + \partial_x (h\mathsf{u}) = 0$$

$$\partial_t (h\mathsf{u}) + \partial_x (h\mathsf{u}^2 + gh^2/2) + gh\partial_x b = h\partial_t \left[\frac{d^2}{3}\partial_{xx}\mathsf{u} + \frac{d}{3}\partial_x d \partial_x \mathsf{u}\right]$$

$$d(x) = h_0 - b(x)$$

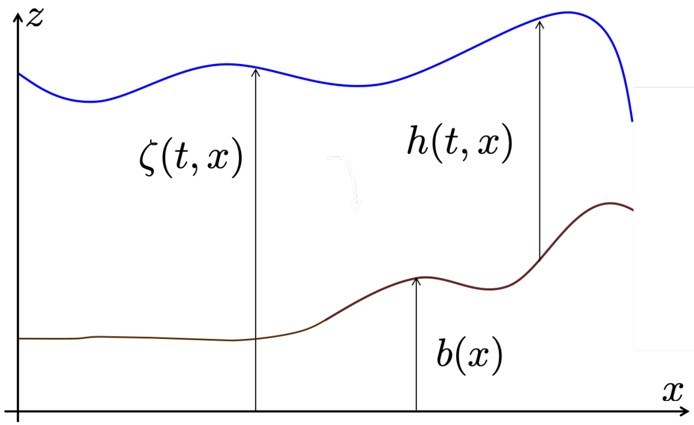
Peregrine, J.Fluid Mech., 1967

$$\partial_t h + \partial_x (h\mathsf{u}) = 0$$

$$\partial_t (h\mathsf{u}) + \partial_x (h\mathsf{u}^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$

$$\mathcal{D} = \partial_t \left[\beta d^2 \partial_{xx}(h\mathsf{u}) + \frac{d}{3}\partial_x d \partial_x (h\mathsf{u})\right] + Bgd^2 \left[d\partial_{xxx}\zeta + 2\partial_x d \partial_{xx}\zeta\right]$$

Madsen & Sorensen, Coast.Eng., 1992



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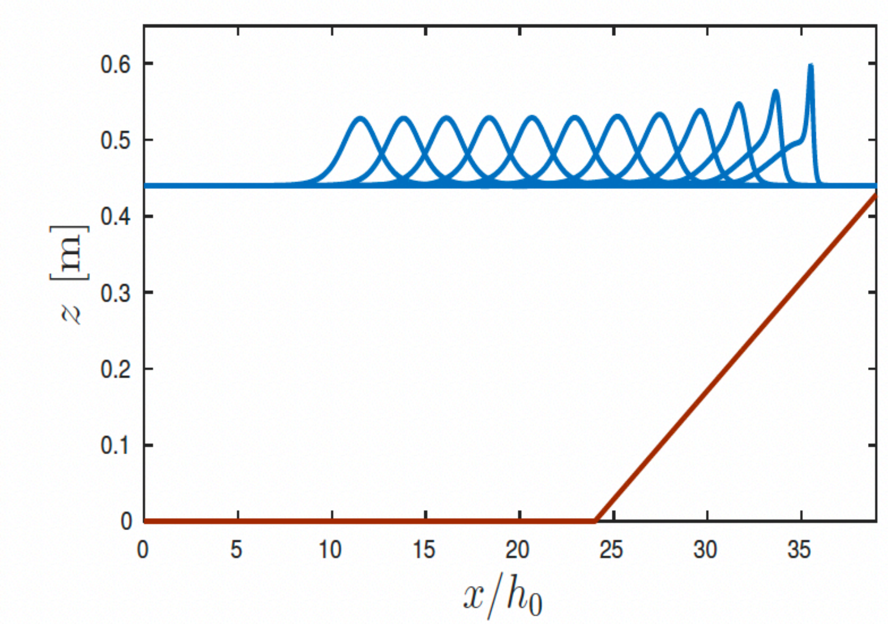
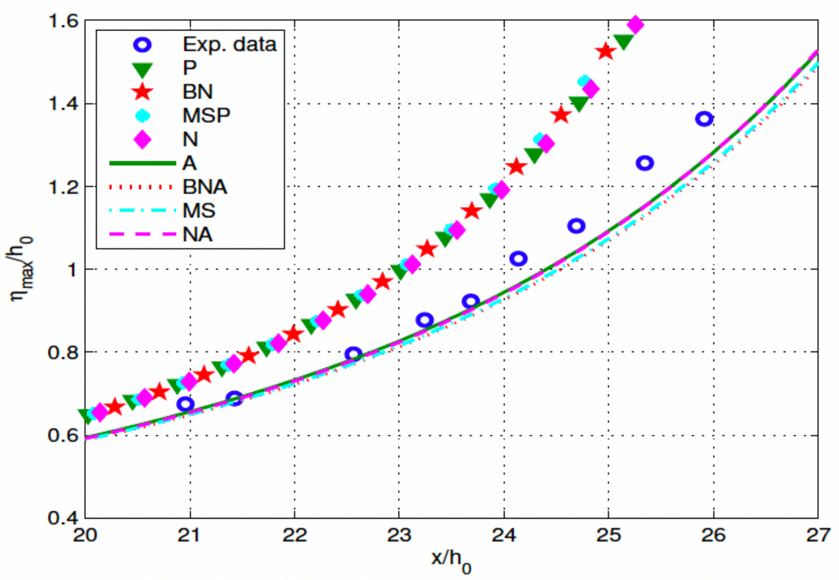
Many variations for a given asymptotic accuracy

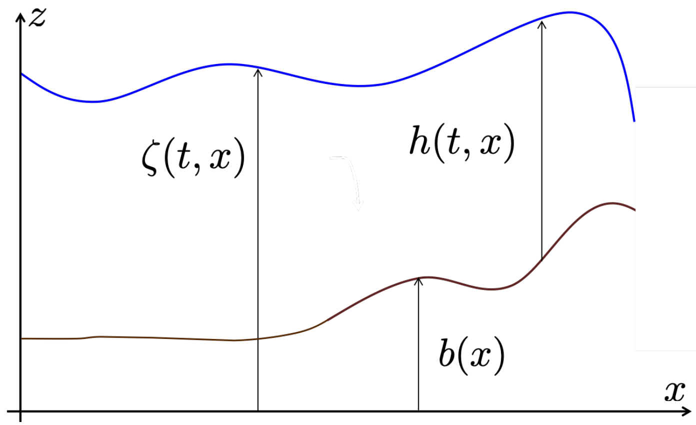
- $\mu^2 d \equiv \mu^2 h$ as the difference is of order $\mu^2\epsilon = \mu^4$
- $\mu^2\partial_{xxt}(d\mathbf{u}) \equiv \mu^2\partial_{xxt}(h\mathbf{u})$ as the difference is of order $\mu^2\epsilon = \mu^4$





Filippini et al, Coast.Eng., 2015

Kazolea & Ricchiuto, J.Hyd.Eng, to appear



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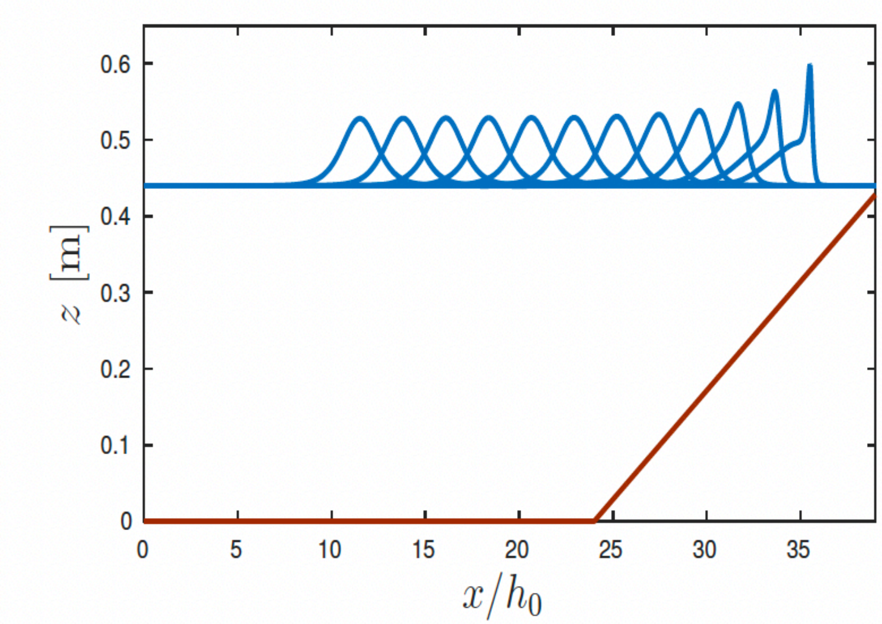
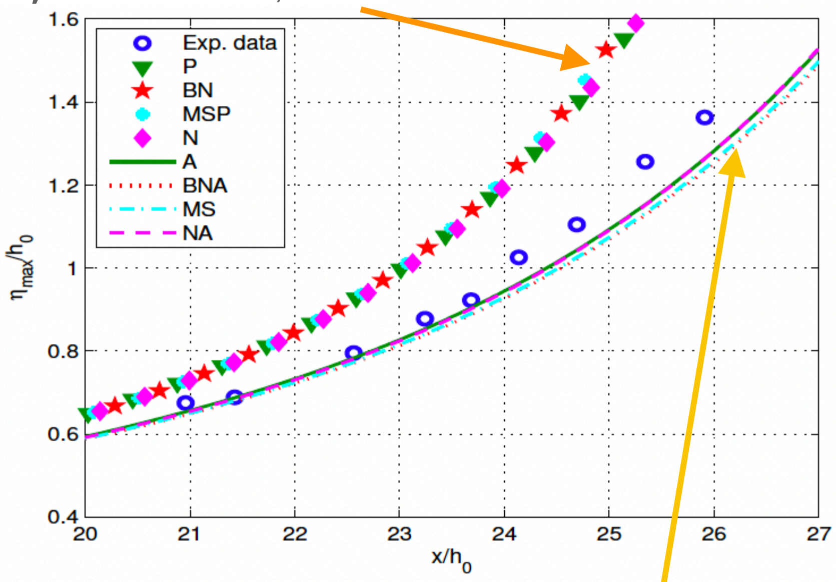
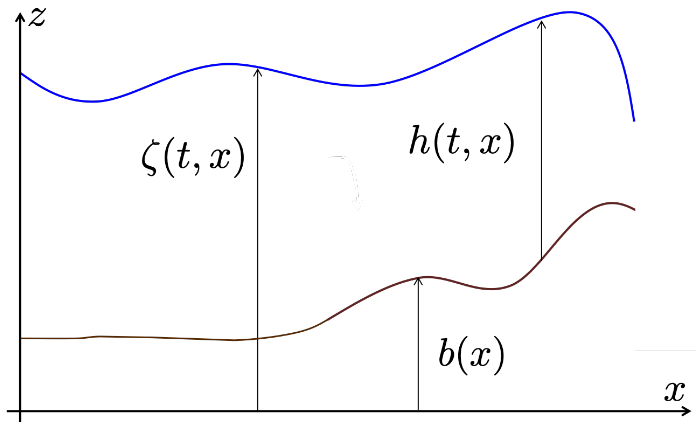


Peregrine, J.Fluid Mech., 1967



Madsen & Sorensen, Coast.Eng., 1992



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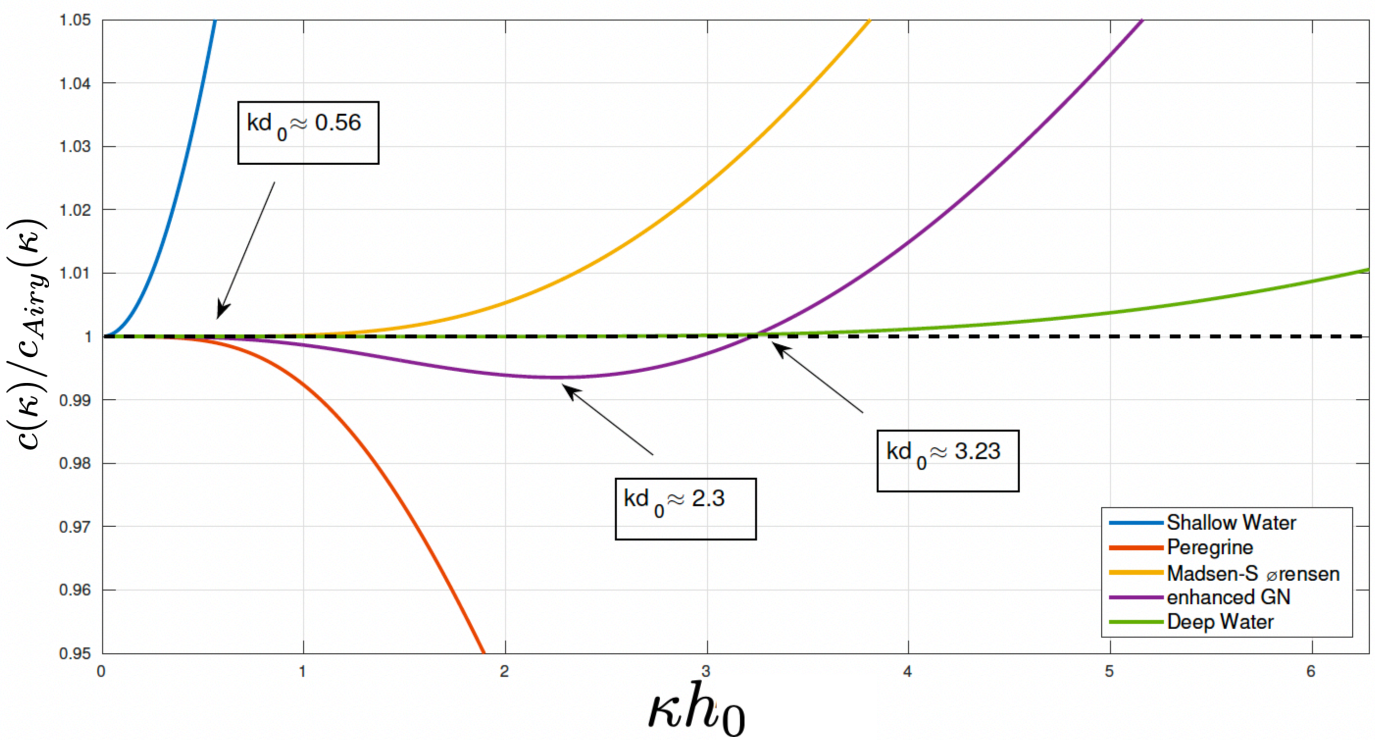

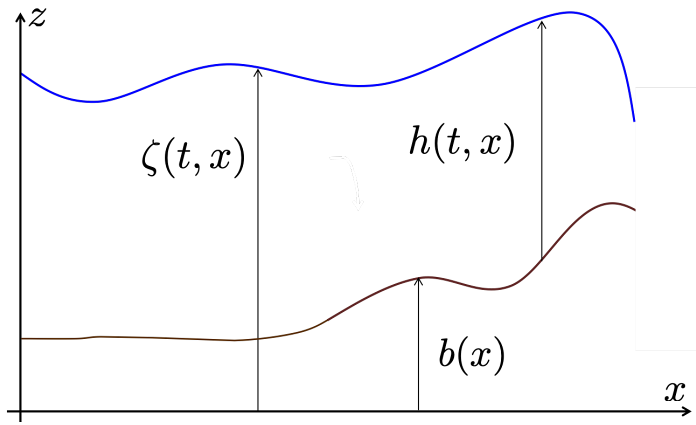

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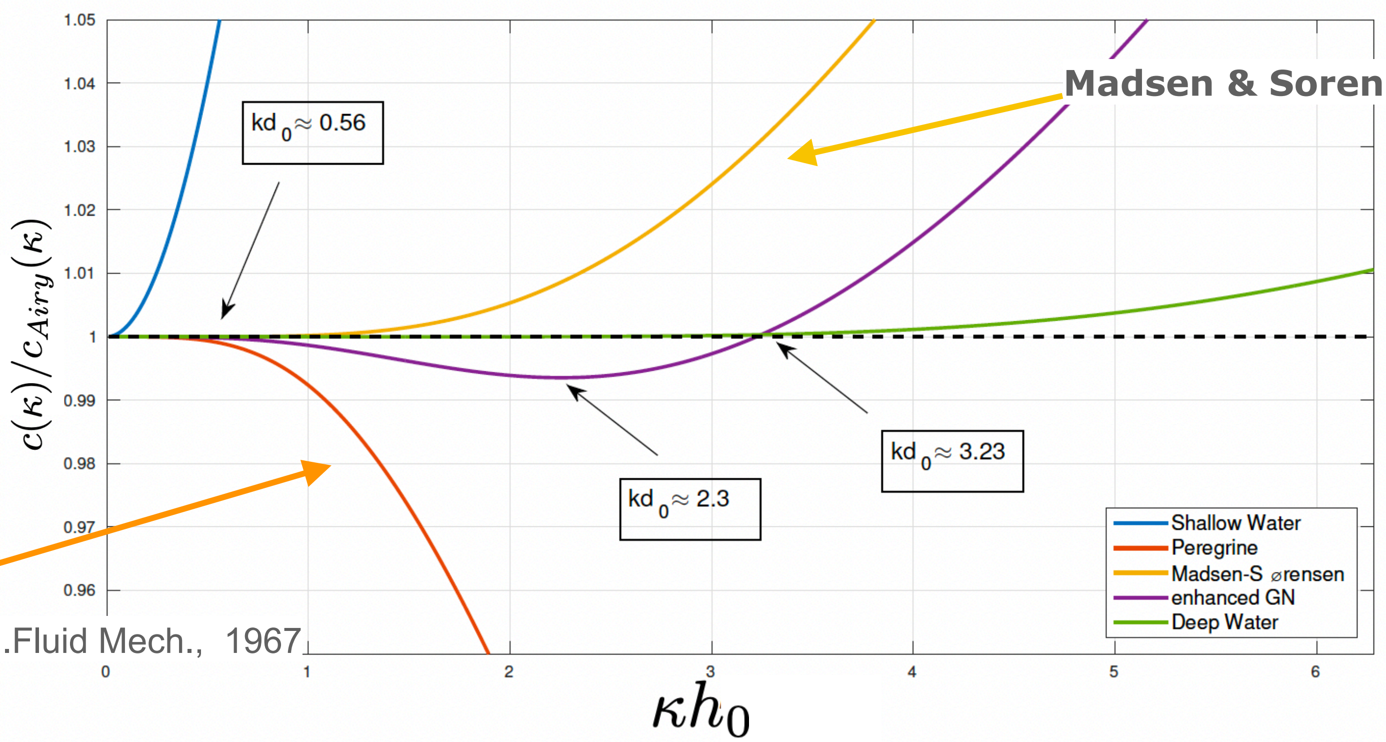
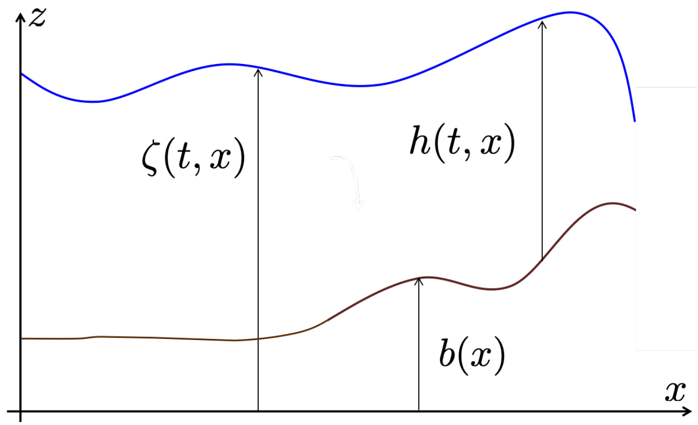
Many variations for a given asymptotic accuracy

* $\mu^2 d \equiv \mu^2 h$ as the difference is of order $\mu^2 \epsilon = \mu^4$
* $\mu^2 \partial_{xxt}(d\mathrm{u}) \equiv \mu^2 \partial_{xxt}(h\mathrm{u})$ as the difference is of order $\mu^2 \epsilon = \mu^4$

Madsen & Sorensen, Coast.Eng., 1992



Peregrine, J.Fluid Mech., 1967



Weakly dispersive (μ^2) corrections

Fully nonlinear: $\epsilon = \mathcal{O}(1)$

$$\partial_t h + \partial_x (h\mathsf{u}) = 0$$

$$\partial_t (h\mathsf{u}) + \partial_x (h\mathsf{u}^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$

$$\mathcal{D} = \alpha \partial_x (h^2 \partial_x \dot{\mathsf{u}}) + (\alpha - 1)\partial_x (h^2 \partial_{xx} \zeta) + \mathcal{Q}(\mathsf{u}, b)$$

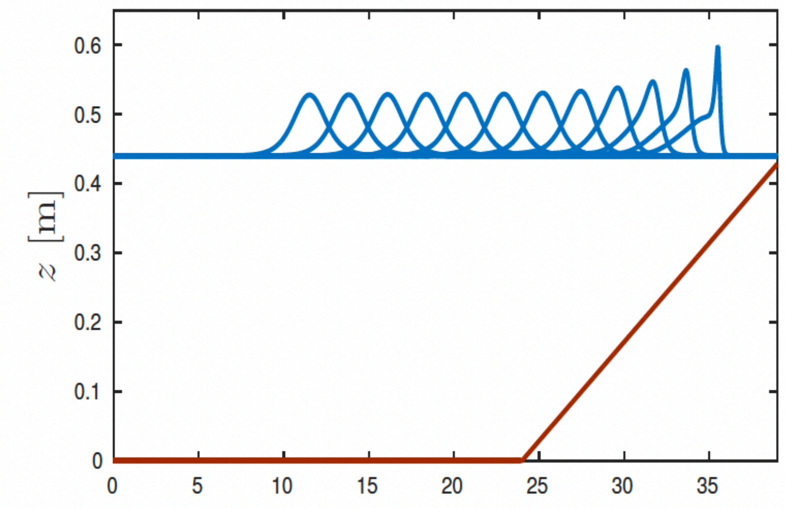
$$\begin{aligned}\mathcal{Q}(\mathsf{u}, b) =& h\partial_x h^2 (\partial_x \mathsf{u})^2 + \frac{2}{3} h^3 \partial_x (\partial_x \mathsf{u})^2 \\ &+ h^2 \partial_x b (\partial_x \mathsf{u})^2 + \frac{h}{2} \partial_{xx} b \partial_x \mathsf{u}^2 + (\partial_x (h^2 \partial_{xx} b) + \partial_x (\partial_x b)^2) \frac{\mathsf{u}^2}{2}\end{aligned}$$

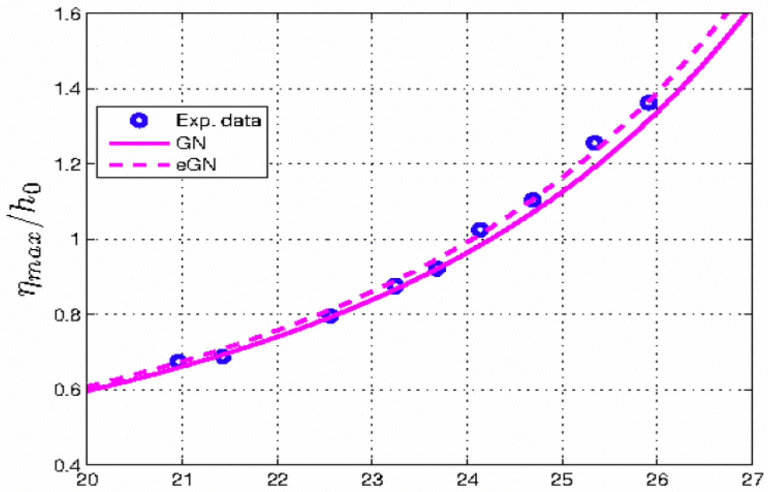
Green & Naghdi, J.Fluid Mech, 1976

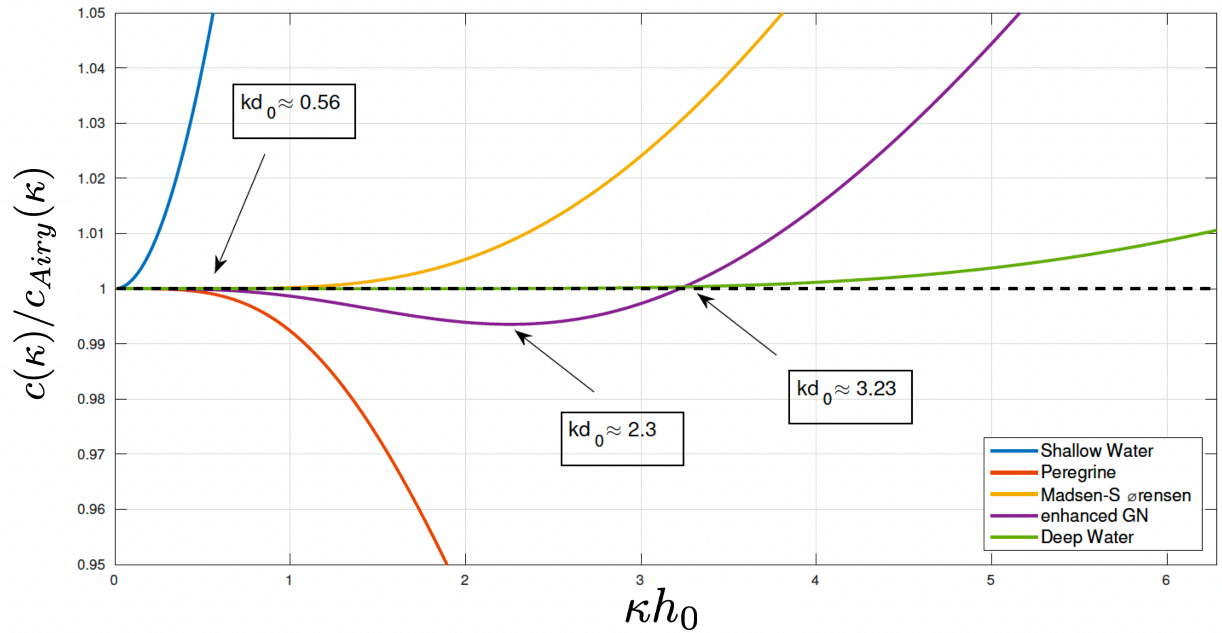
Chazel et al, J.Sci.Comp. 2011

Weakly dispersive (μ^2) corrections

Fully nonlinear: $\epsilon = \mathcal{O}(1)$







MultiD numerical approximation

	Weakly nonlinear Weakly dispersive	Fully nonlinear Fully dispersive
Structured Grids	● **Madsen et al**, 1992 ● **Nwogu**, 1994 ● **Beji & Nadaoka**, 1996 ● etc. etc.	● **Wei & Kirby**, 1995 ● **Shi et al**, 2012 ● **Lannes & Marche**, 2015 ● etc. etc.
Unstructured Grids	○ **Walkley & Berzins**, 2002 ○ **Eskilsson & Sherwin**, 2006 ○ **Kazolea et al**, 2012 ○ **Ricchiuto & Filippini**, 2014 ○ etc. etc.	☑ **Filippini et al**, 2017 ☑ **Duran & Marche**, 2017 ☑ **Assiouene et al**, 2020 ☑ **Busto et al**, 2021 ☑ etc. etc.

Certainly forgetting someone here …

Enhanced Serre-Green-Naghdi equations in multiD

$$\partial_t h + \partial_x (h\mathsf{u}) = 0$$

$$\partial_t (h\mathsf{u}) + \partial_x (h\mathsf{u}^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$

$$\mathcal{D} = \alpha\partial_x (h^2\partial_x \dot{\mathsf{u}}) + (\alpha - 1)\partial_x (h^2 \partial_{xx}\zeta) + \mathcal{Q}(\mathsf{u}, b)$$

$$\begin{aligned}\mathcal{Q}(\mathsf{u}, b) =& h\partial_x h^2 (\partial_x \mathsf{u})^2 + \frac{2}{3} h^3 \partial_x (\partial_x \mathsf{u})^2 \\ &+ h^2 \partial_x b(\partial_x \mathsf{u})^2 + \frac{h}{2}\partial_{xx} b \partial_x \mathsf{u}^2 + (\partial_x (h^2 \partial_{xx} b) + \partial_x (\partial_x b)^2)\frac{\mathsf{u}^2}{2}\end{aligned}$$

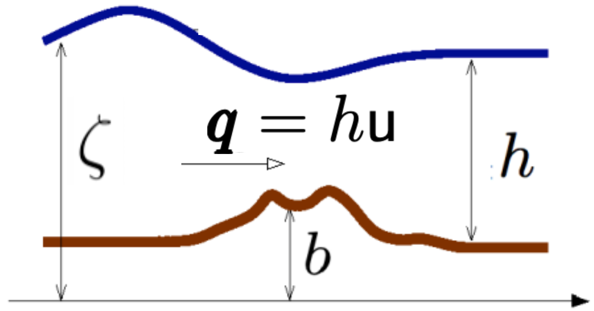
Enhanced Serre-Green-Naghdi equations in multiD

$$\partial_t h + \partial_x(h\mathsf{u}) = 0$$

$$\partial_t(h\mathsf{u}) + \partial_x(h\mathsf{u}^2 + gh^2/2) + gh\partial_x b = \varphi$$

$$\varphi - \alpha\partial_x(h\partial_x\varphi - \varphi\partial_x h) = -\partial_x(h^2\partial_{xx}\zeta) + \mathcal{Q}$$

$$\begin{aligned}\mathcal{Q}(\mathsf{u}, b) =& h\partial_x h^2(\partial_x\mathsf{u})^2 + \frac{2}{3}h^3\partial_x(\partial_x\mathsf{u})^2 \\ &+ h^2\partial_x b(\partial_x\mathsf{u})^2 + \frac{h}{2}\partial_{xx}b\partial_x\mathsf{u}^2 + (\partial_x(h^2\partial_{xx}b) + \partial_x(\partial_x b)^2)\frac{\mathsf{u}^2}{2}\end{aligned}$$



Enhanced Serre-Green-Naghdi equations in multiD

$$\partial_t h + \nabla \cdot \boldsymbol{q} = 0$$

$$\partial_t \boldsymbol{q} + \nabla \cdot \left(\frac{\boldsymbol{q} \otimes \boldsymbol{q}}{h} \right) + gh\nabla\zeta - \boldsymbol{\varphi} = 0$$

Hyperbolic step

$$\boldsymbol{\varphi} + \alpha \mathrm{T}_h(\boldsymbol{\varphi}) = \mathcal{R}(h, \boldsymbol{q}, b)$$

$$\mathcal{R}(h, \boldsymbol{q}, b) = \mathrm{T}_h(h\nabla\zeta) + \mathcal{Q}\left(\frac{\boldsymbol{q}}{h}\right)$$

Elliptic step

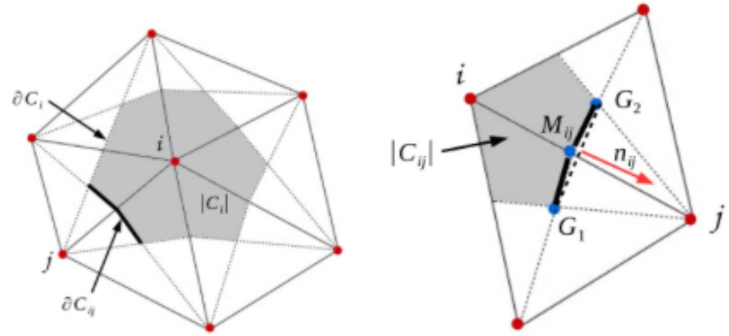
For constant bathymetry

$$\mathrm{T}_h(\boldsymbol{\varphi}) = -\nabla(h\nabla \cdot \boldsymbol{\varphi}) + \nabla(\boldsymbol{\varphi} \cdot \nabla h)$$

Filippini et al, J.Comput.Phys.2016

Kazolea et al, Ocean Mod., 2023

Enhanced Serre-Green-Naghdi solver



$$\partial_t h + \nabla \cdot \boldsymbol{q} = 0$$

$$\partial_t \boldsymbol{q} + \nabla \cdot \left(\frac{\boldsymbol{q} \otimes \boldsymbol{q}}{h} \right) + gh\nabla\zeta - \boldsymbol{\varphi} = 0$$

Hyperbolic step

Nodal Finite volume

Well-balanced Roe or HLL numerical fluxes/sources

Explicit high order time stepping (Runge-Kutta SSP3)

Compact nodal k-th derivative recovery via iterative corrected Green-Gauss*

Kazolea et al, Ocean Mod., 2023

Enhanced Serre-Green-Naghdi solver

$$\boldsymbol{\varphi} + \alpha \mathrm{T}_h(\boldsymbol{\varphi}) = \mathcal{R}(h, \boldsymbol{q}, b)$$

$$\mathcal{R}(h, \boldsymbol{q}, b) = \mathrm{T}_h(h\nabla\zeta) + \mathcal{Q}\left(\frac{\boldsymbol{q}}{h}\right)$$

Elliptic step

For contant bathymetry

$$\mathrm{T}_h(\boldsymbol{\varphi}) = -\nabla(h\nabla\cdot\boldsymbol{\varphi}) + \nabla(\boldsymbol{\varphi}\cdot\nabla h)$$

We solve it with standard H1 linear finite elements (not in H(div) …) :

- Block SPD structure
- H1 is not the natural space

—> spurious ``curl modes'' need stabilization/damping

Enhanced Serre-Green-Naghdi solver

On the stability of I - grad div :

- H(div) conforming FE space

- Stability in H1
  - Curl stabilization: H1=H(div)+H(curl) **Costabel**, J.Math.Anal.Appl. 1991

    Bonnet-Ben Dhia et al, CRAS 2001, **Bonnet-Ben Dhia et al**, J. Comput. Appl. Math. 2007

  - Mixed form + stabilization, **Bonito et al**, M2NA 2016, **Chabassier & Duruflé** 2018

  - Laplacian stabilization

    Mardal et al, SINUM 2002

Enhanced Serre-Green-Naghdi solver

On the stability of I - grad div :

$$\boldsymbol{\varphi} = (I + \alpha \mathrm{T}_h)^{-1} \mathcal{R}(h^n, \boldsymbol{q}^n, b)$$

$$\frac{h^{n+1} - h^n}{\Delta t} + \nabla \cdot \hat{\boldsymbol{q}}^n = \nabla \cdot (D_h \widehat{\nabla h}^n)$$

$$\frac{\boldsymbol{q}^{n+1} - \boldsymbol{q}^n}{\Delta t} + \nabla \cdot \hat{\boldsymbol{F}}^n_{\boldsymbol{q}} + \boldsymbol{S}^n_b - \boldsymbol{\varphi}^n = \nabla \cdot (D_{\boldsymbol{q}} \widehat{\nabla \boldsymbol{q}}^n)$$

- Laplacian stabilization

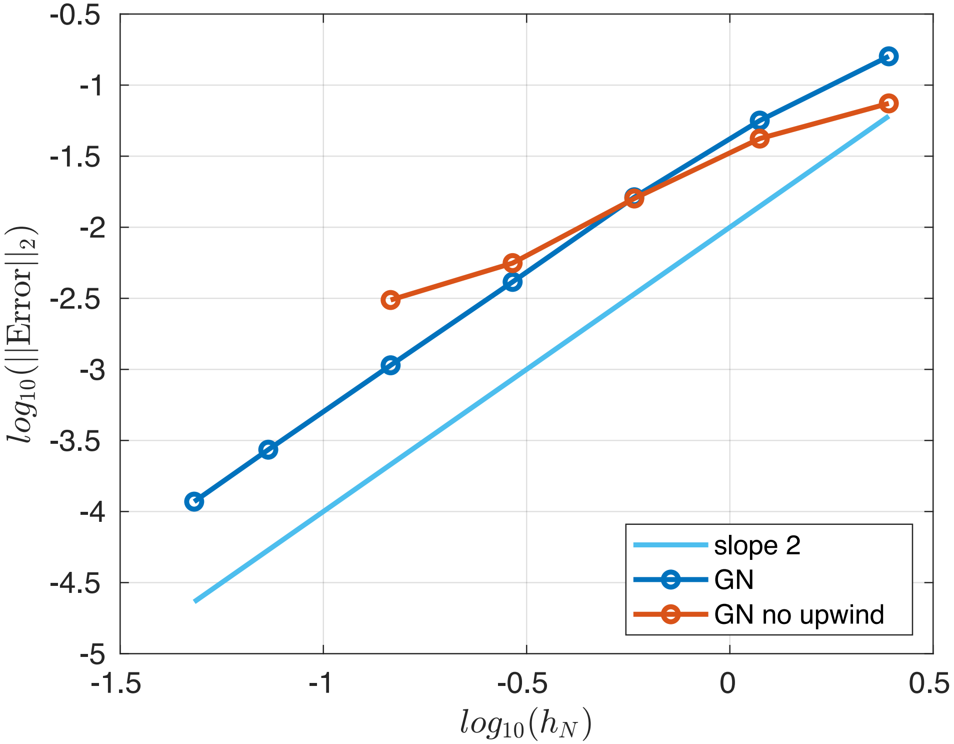
  Mardal et al, SINUM 2002

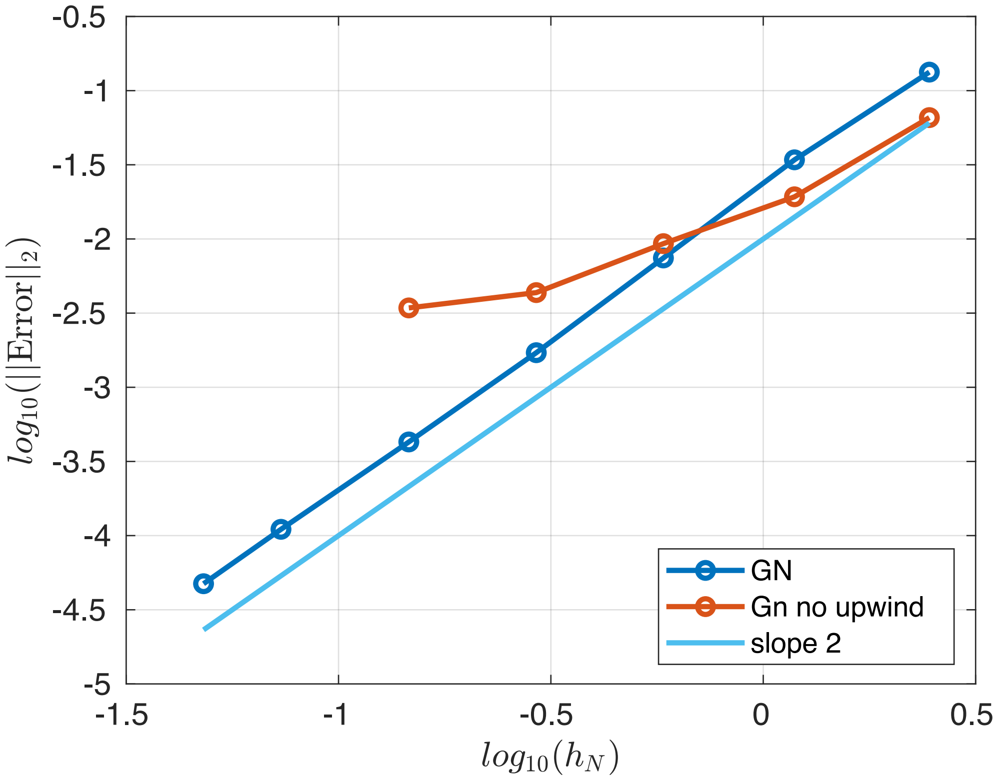
Embedded in FV numerical fluxes

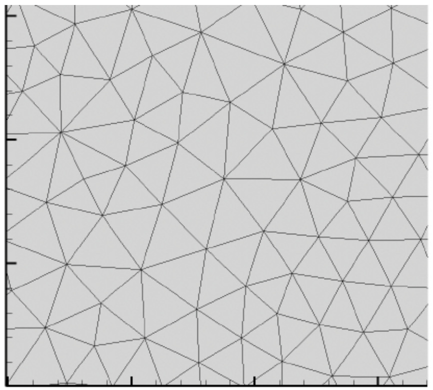
Parabolic damping of spurious curl modes

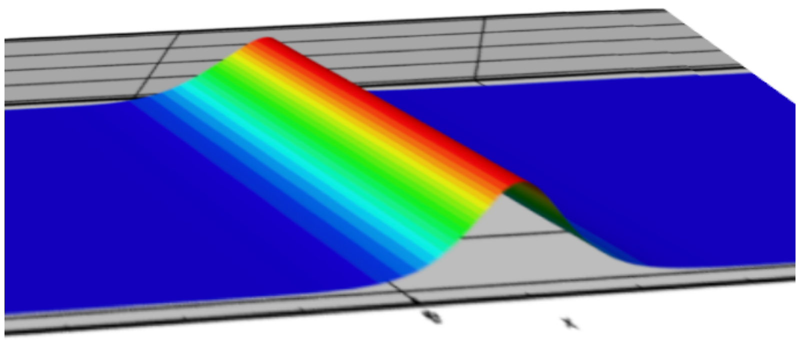
Kazolea et al, Ocean Mod., 2023

Enhanced Serre-Green-Naghdi solver









Analytical solitary wave
2d convergence
Unstructured triangulations

Enhanced Serre-Green-Naghdi solver

The formal consistency of the method is

$$\partial_t h + \nabla \cdot \boldsymbol{q} = \mathcal{O}(\Delta x^3)$$

$$(I + \alpha \mathrm{T}_h)(\partial_t \boldsymbol{q} + \nabla \cdot F_{\boldsymbol{q}} + \boldsymbol{S}_b) = \mathcal{R} + \mathcal{O}(\mu^2 \Delta x^2)$$

When using third order polynomial reconstruction and RK3

Kazolea et al, Ocean Mod., 2023

Enhanced Serre-Green-Naghdi solver

The formal consistency of the method is

$$\partial_t h + \nabla \cdot \boldsymbol{q} = \mathcal{O}(\Delta x^3)$$

$$(I + \alpha \mathrm{T}_h)(\partial_t \boldsymbol{q} + \nabla \cdot F_{\boldsymbol{q}} + \boldsymbol{S}_b) = \mathcal{R} + \mathcal{O}(\mu^2 \Delta x^2)$$

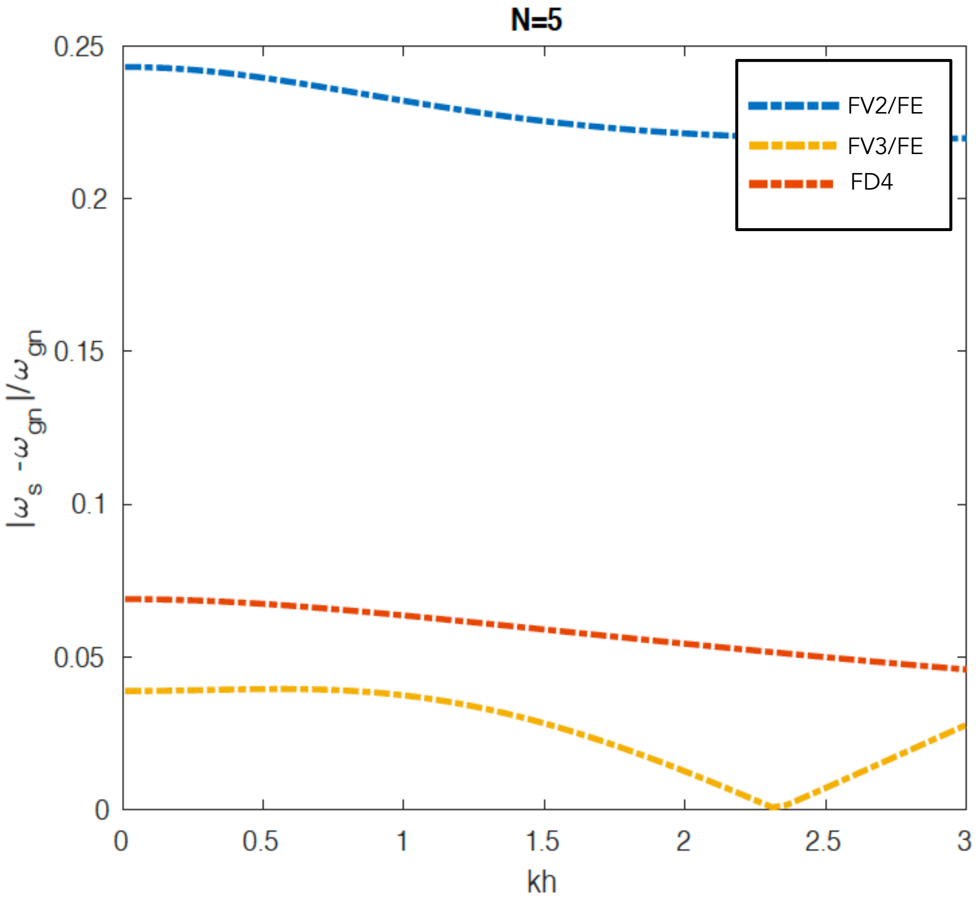
When using third order polynomial reconstruction and RK3

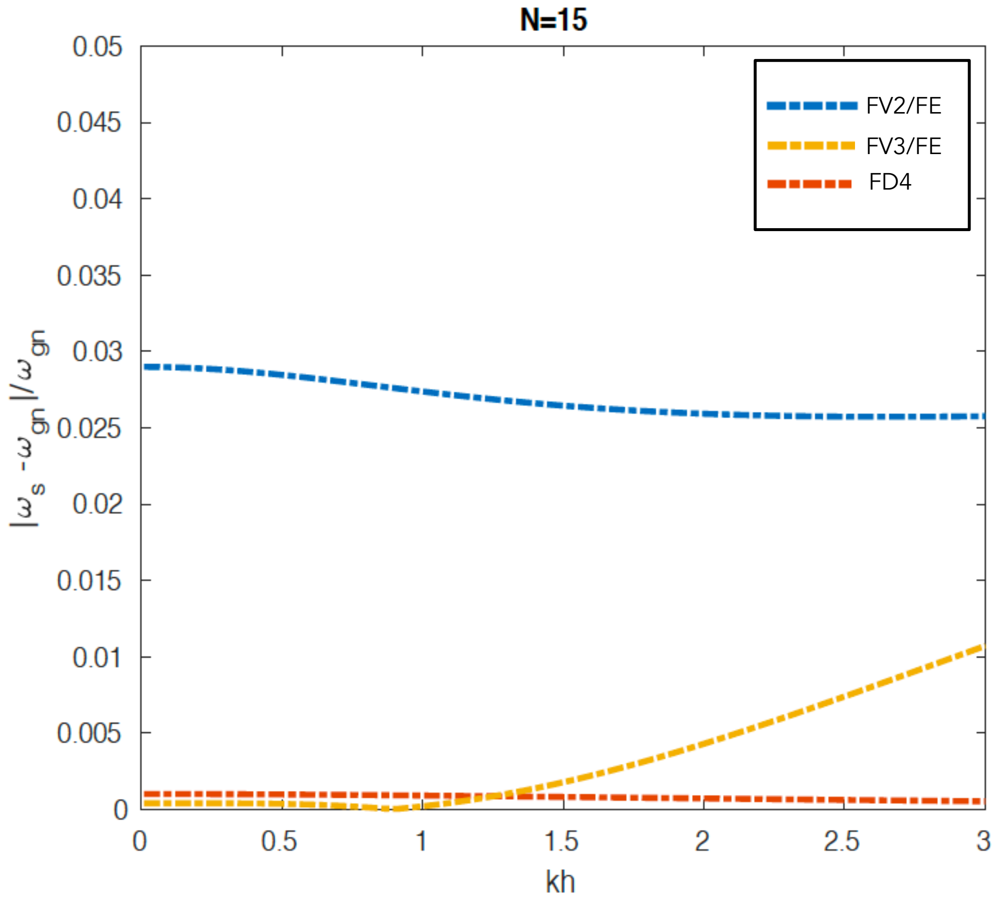
What is the effect of this ?

Kazolea et al, Ocean Mod., 2023

Enhanced Serre-Green-Naghdi solver

Dispersion error (time continuous): $c(\kappa)_{\text{num}} - c(\kappa)_{\text{GN}}$



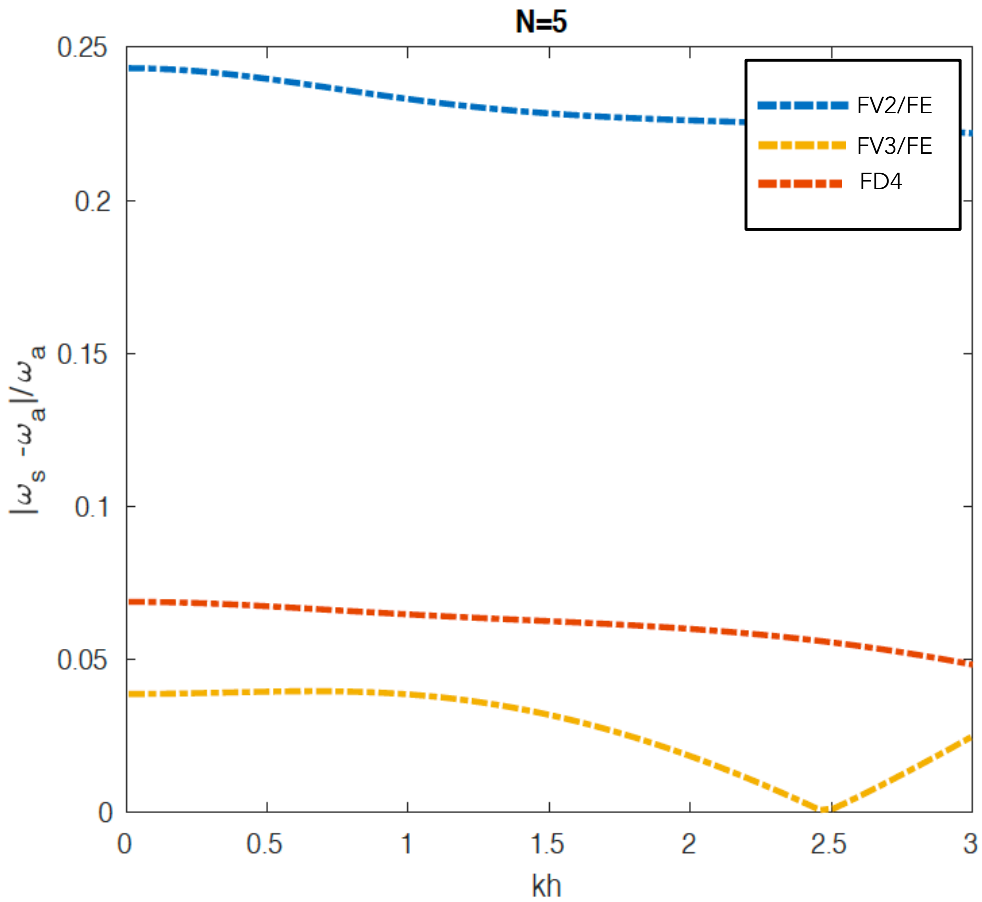


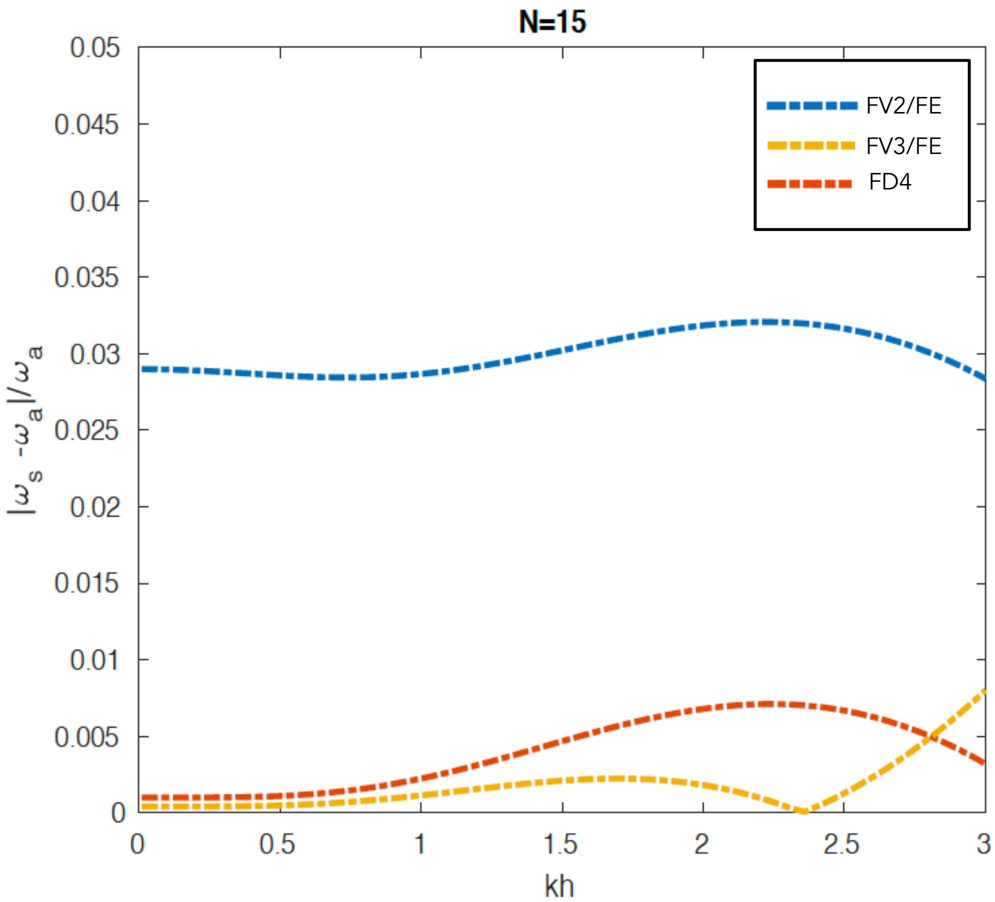
Filippini et al, J.Comput.Phys.2016

Kazolea et al, Ocean Mod., 2023

Enhanced Serre-Green-Naghdi solver

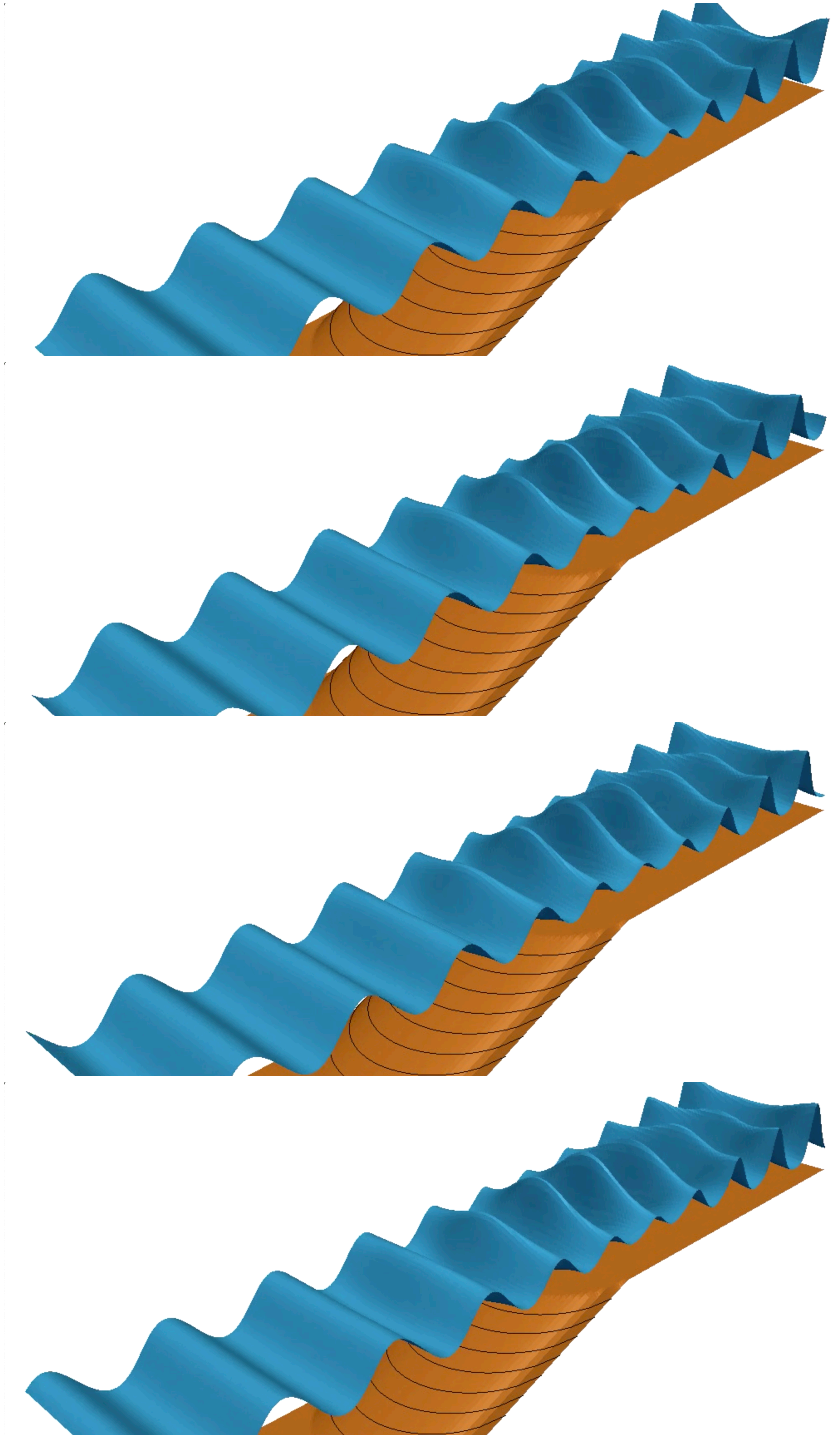
Dispersion error (time continuous): $c(\kappa)_{\text{num}} - c(\kappa)_{\text{Euler}}$

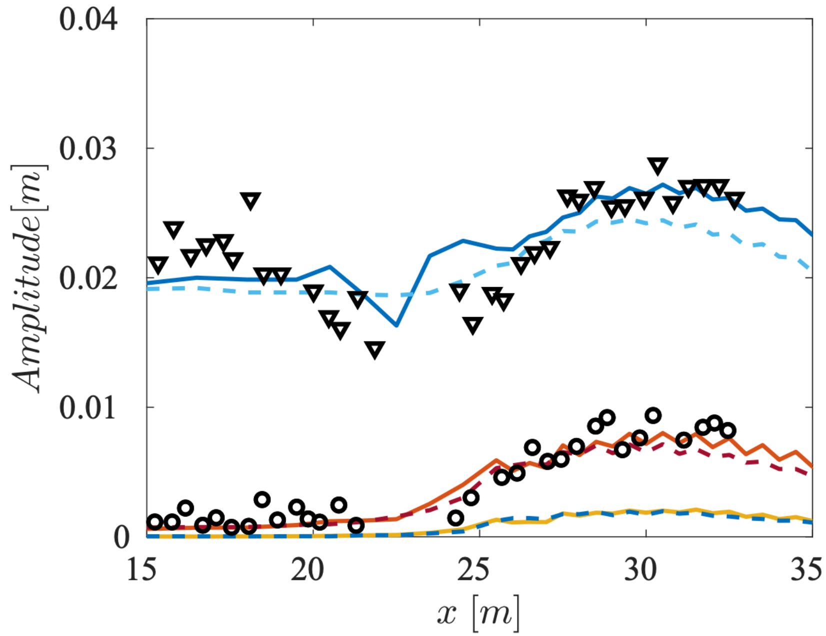


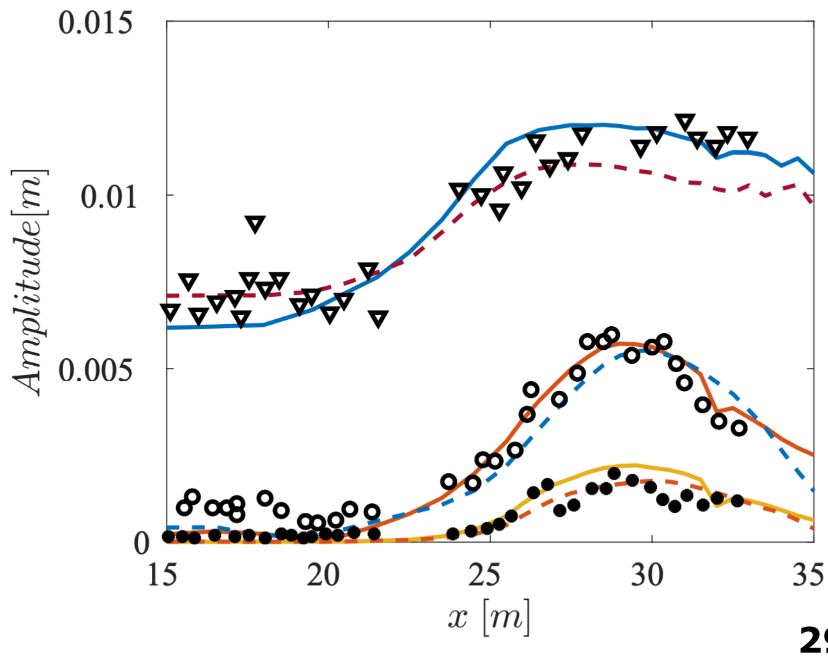


Filippini et al, J.Comput.Phys.2016

Kazolea et al, Ocean Mod., 2023

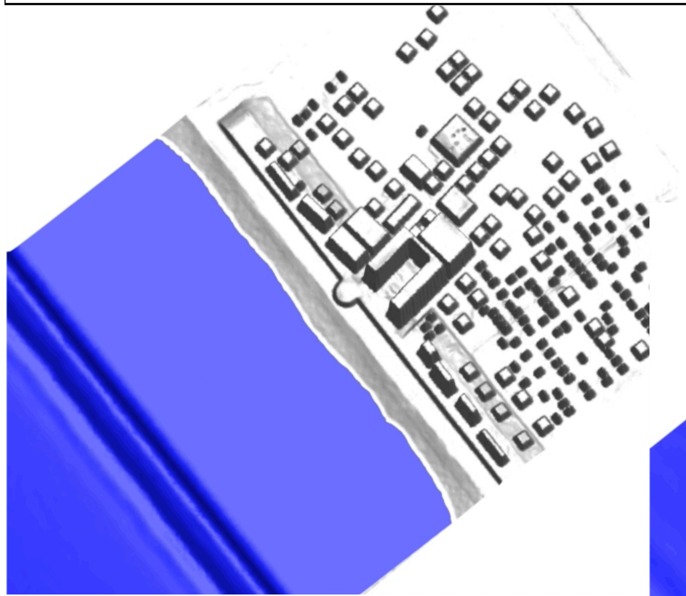


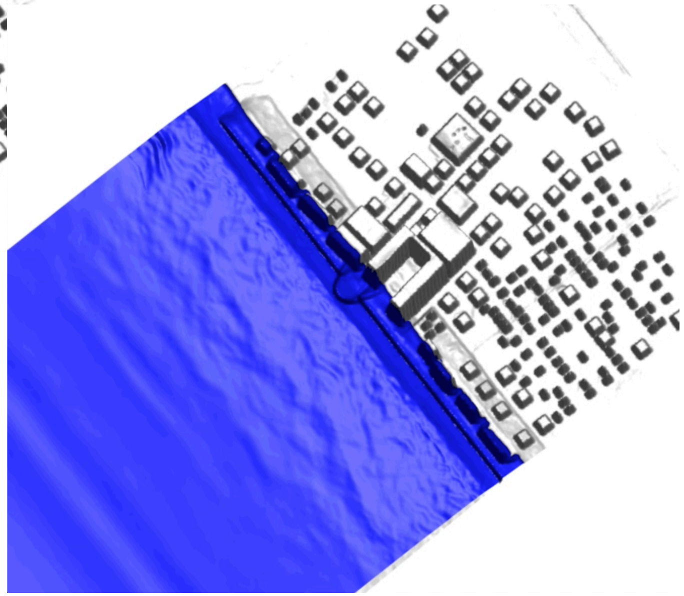

0.04
0.03
0.02
0.01
0
15
20
25
30
35
$Amplitude\,[m]$
$x\;[m]$

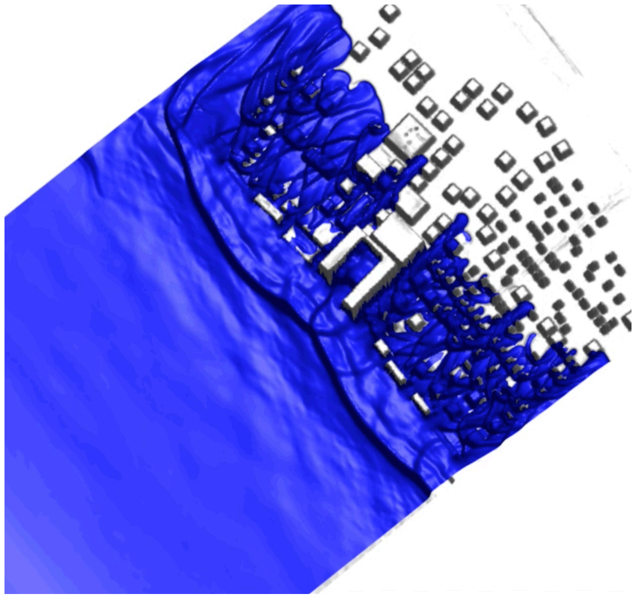

0.015
0.01
0.005
0
15
20
25
30
35
$Amplitude\,[m]$
$x\;[m]$

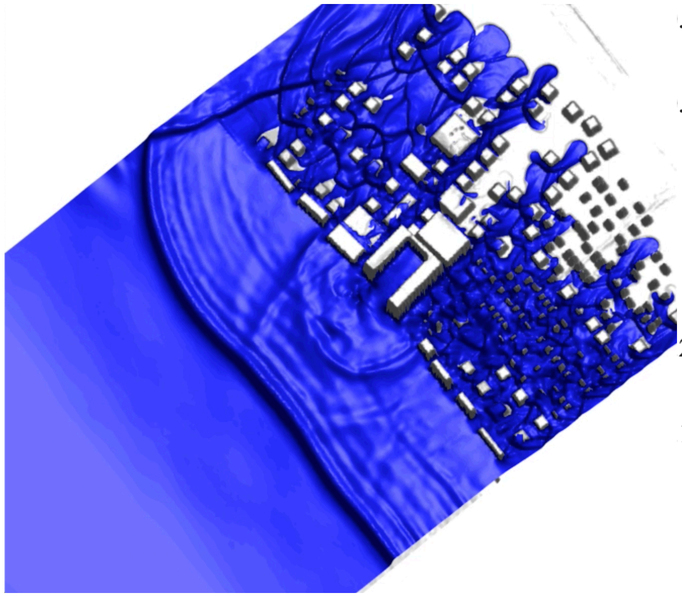
MultiD numerical approximation

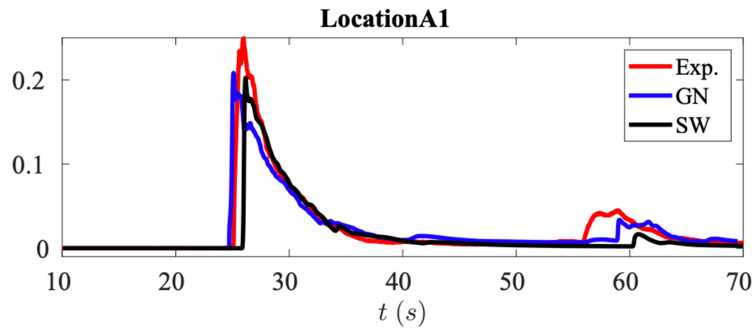
Validation

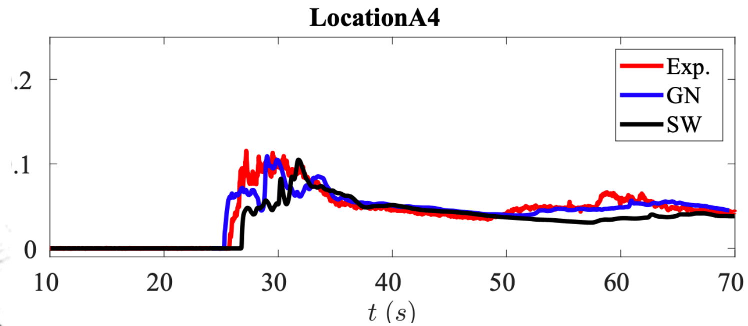


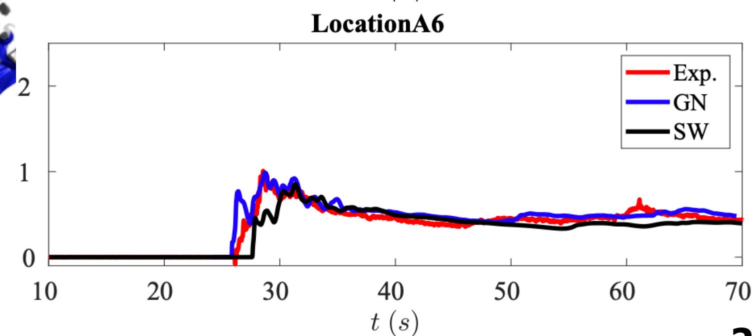








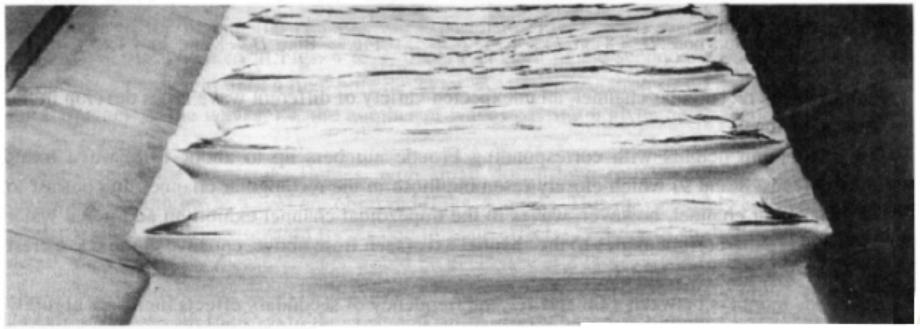




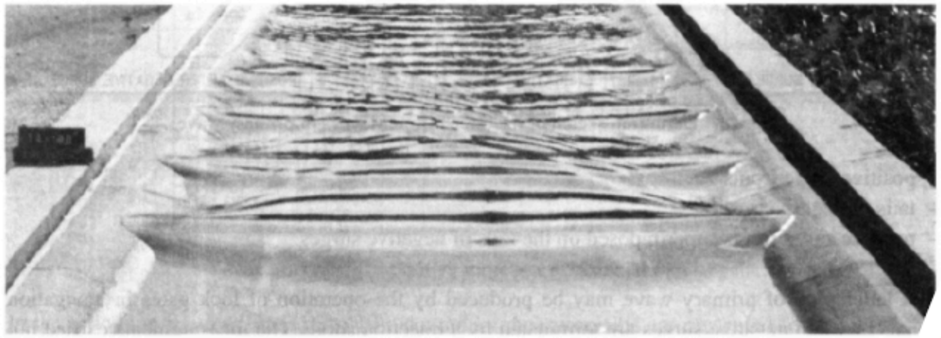
Undular bores simulations

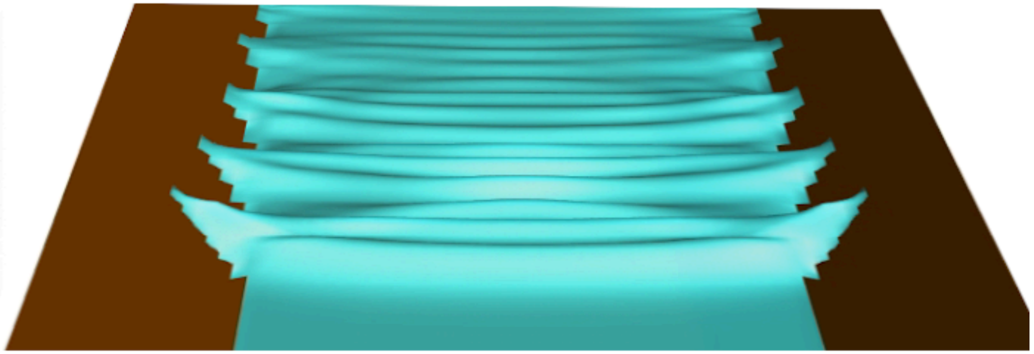
Undular bore simulations | Treske experiments

$Fr = 1.10$



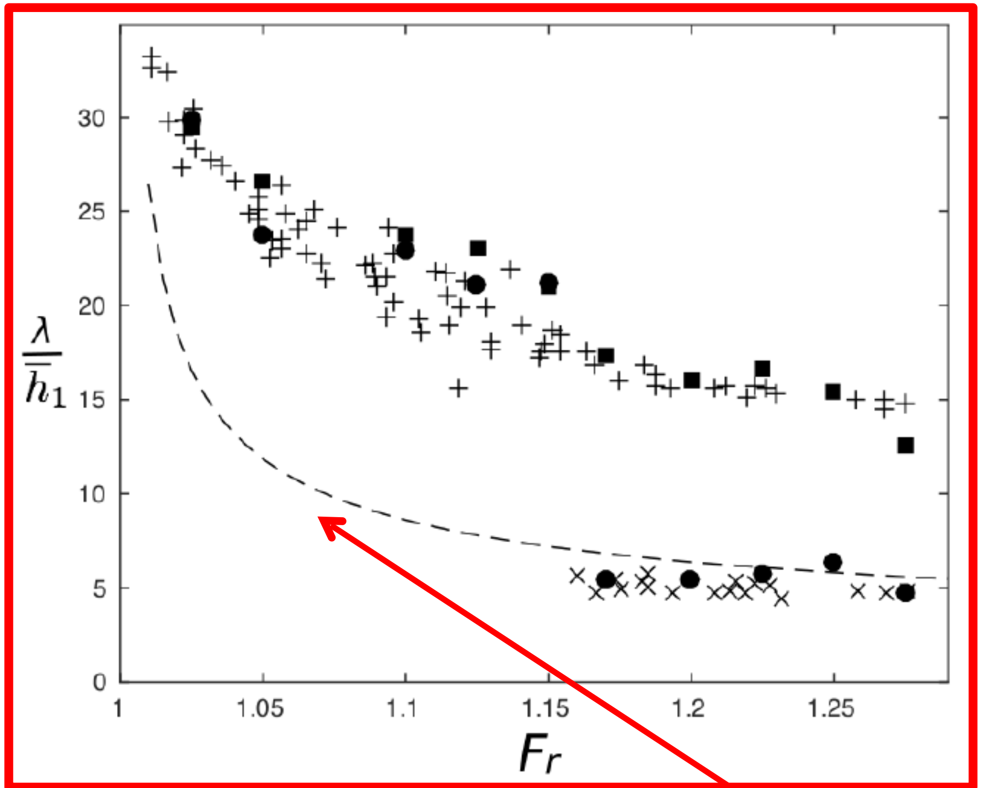
$Fr = 1.17$



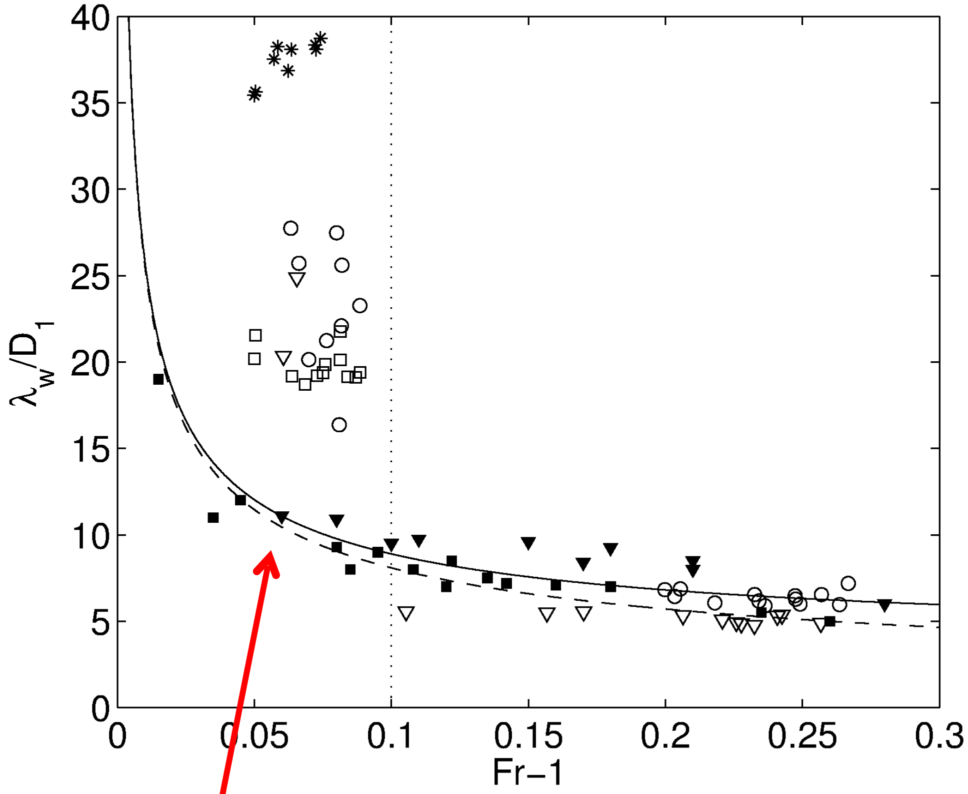


Undular bore simulations | Treske experiments

Treske, *J. Hydraulic Research, 1994*



Bonneton et al, *J. Geophysical Research - Oceans, 2015*

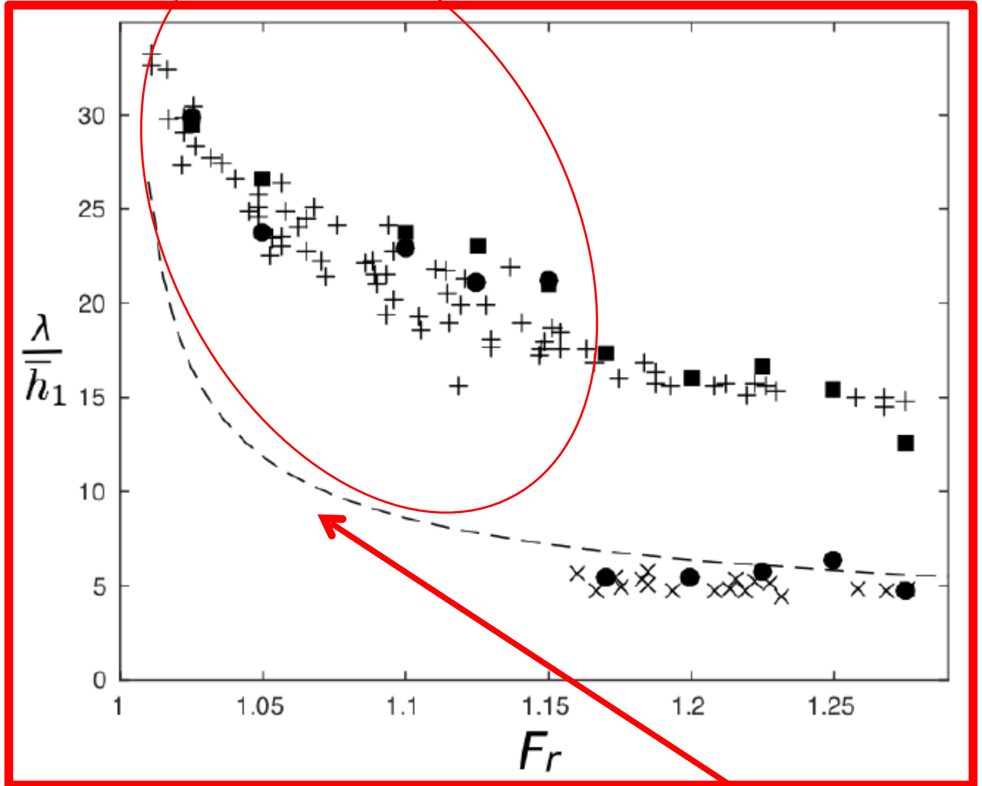


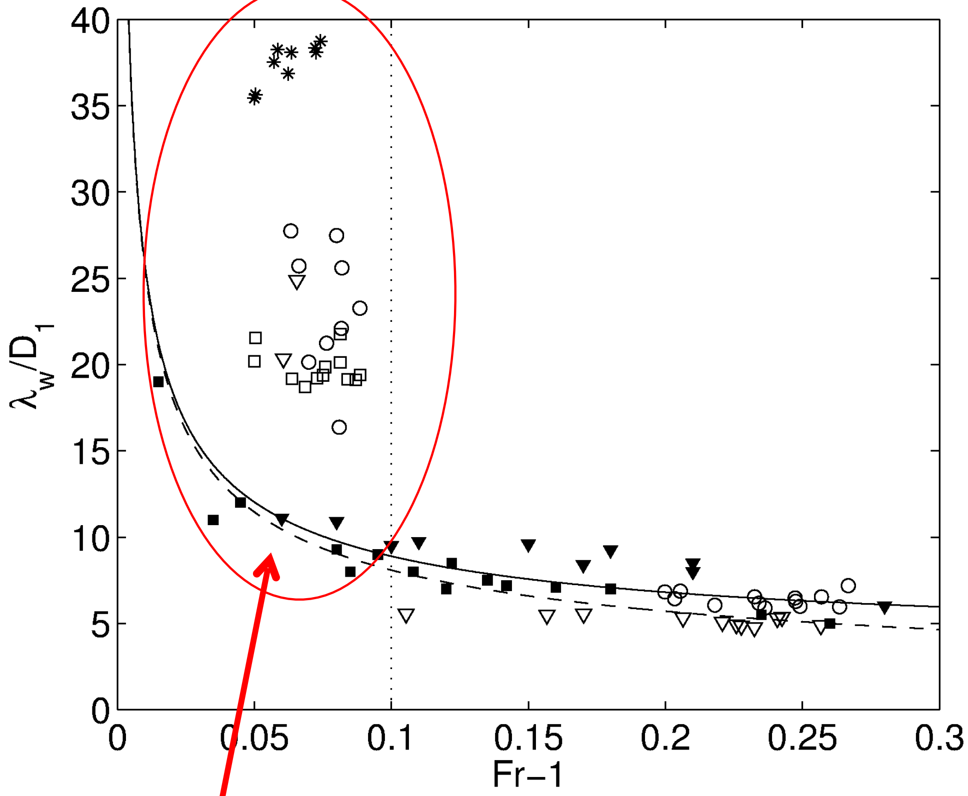
Comparison with data: transition from low Froude to high Froude regime

Lemoine theory for Favre waves

Treske, *J. Hydraulic Research, 1994*

Bonneton et al, *J. Geophysical Research - Oceans, 2015*





Comparison with data: transition from low Froude to high Froude regime

Lemoine theory for Favre waves

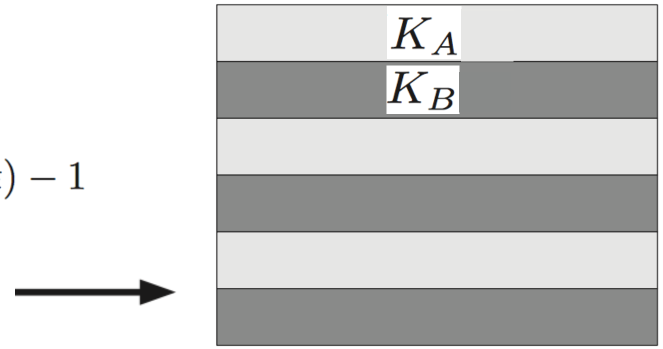
Asymptotic analysis

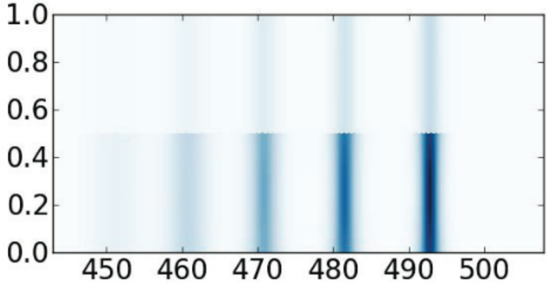
Several elements suggest that it may be an hydrostatic phenomenon:

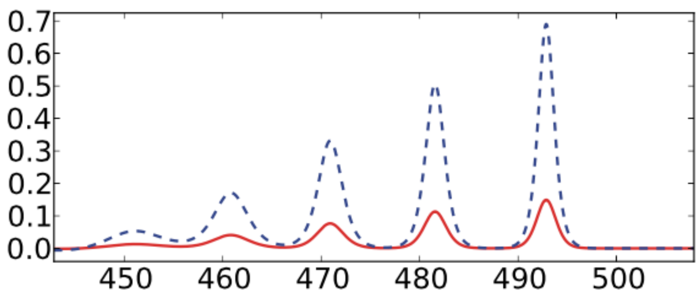
- It is predominant on the banks (very shallow limit)
- It involves long(er) waves
- Dispersion in wave propagation in heterogenous media

Ketcheson & Quessada de Luna, Multiscale Mod. Simul., 2015

$$\epsilon_{tt} - \nabla \cdot \left(\frac{1}{\rho(\mathbf{x})} \nabla \sigma(\epsilon, \mathbf{x}) \right) = 0, \;\; \sigma(\epsilon, \mathbf{x}) = \exp(K(\mathbf{x})\epsilon) - 1$$







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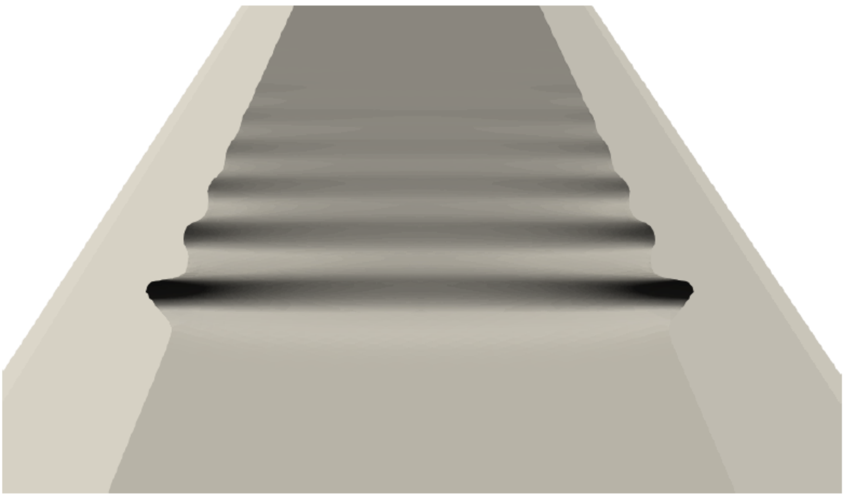
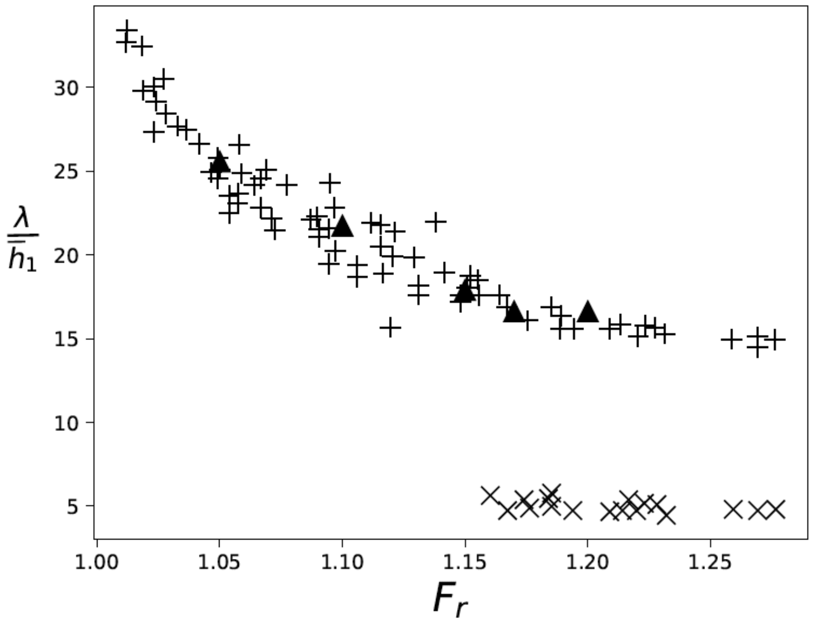
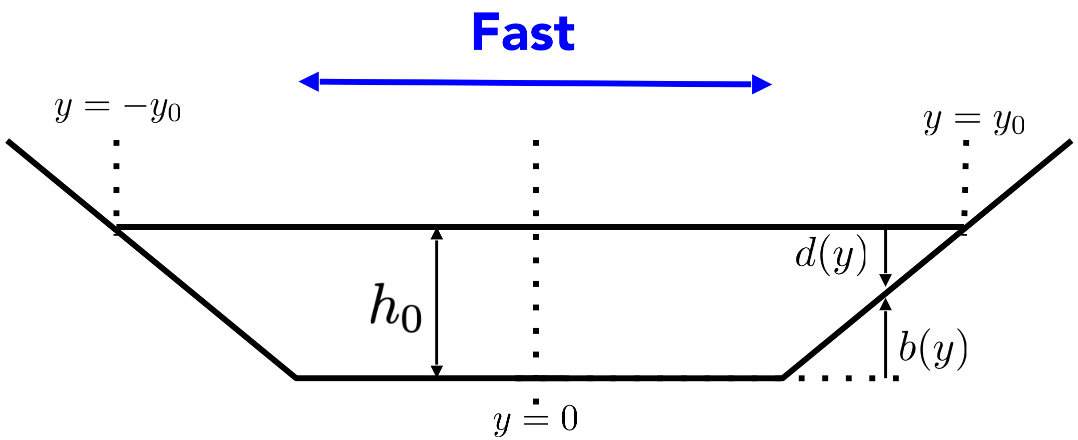
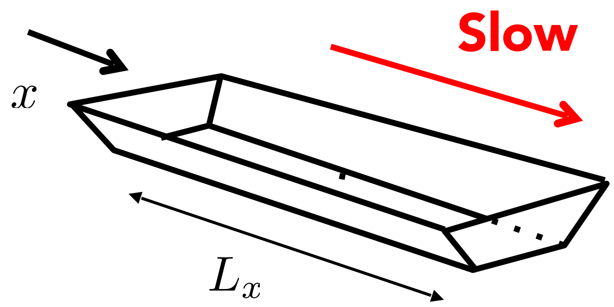




Shallow water simulations !

- Shallow water waves (hydrostatic, no dispersion terms)
- Linear waves
- Scale separation between transverse (fast) and longitudinal (slow) waves

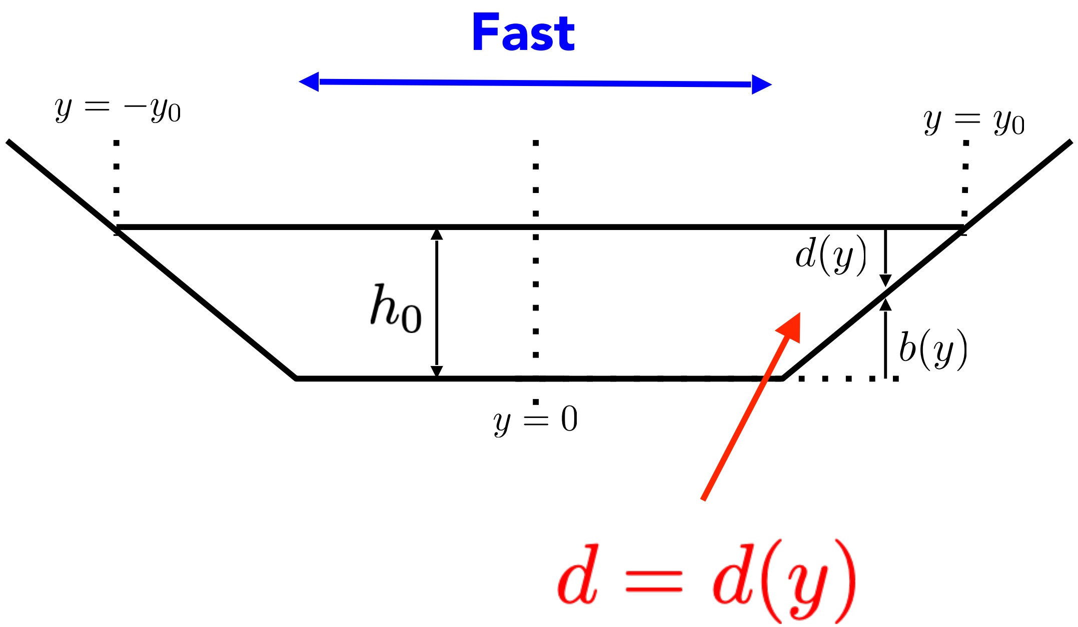


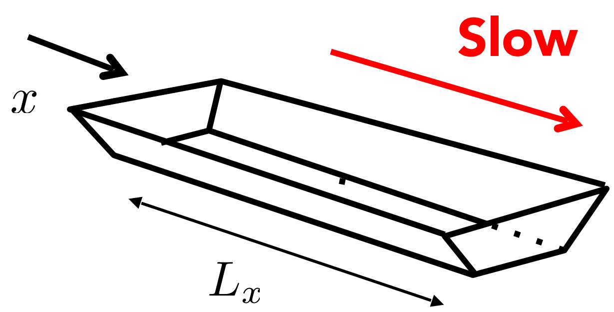


$$\tau_y \ll \tau_x$$

$$L_y = \tau_y \sqrt{g h_0} \ll L_x$$

$$\delta = \frac{L_y}{L_x} = \frac{\tau_y}{\tau_x} \ll 1$$



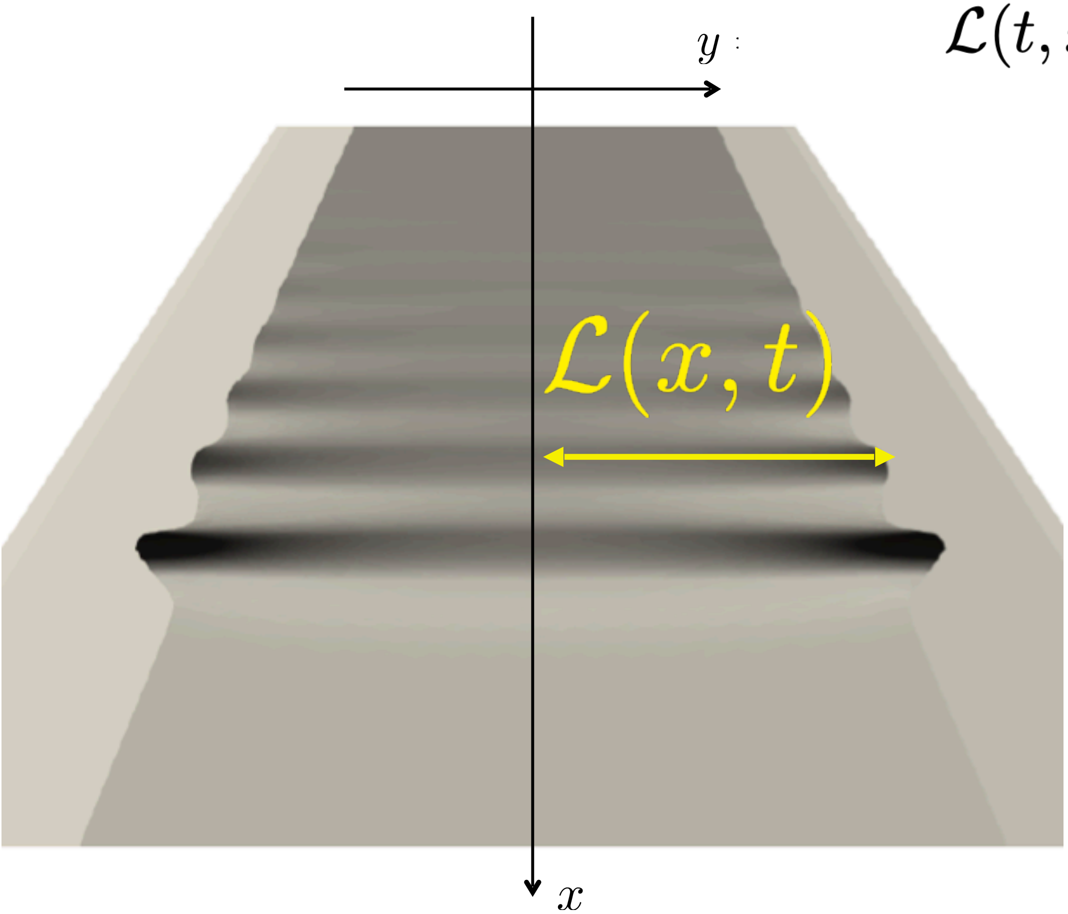


2D NLSW in dimensionless form

$$\partial_t \zeta + \partial_x((d+\epsilon\zeta)u) + \frac{1}{\delta}\partial_y((d+\epsilon\zeta)v) = 0$$

$$\partial_t u + \epsilon u \partial_x u + \frac{\epsilon}{\delta} v \partial_y u + \partial_x \zeta = 0$$

$$\partial_t v + \epsilon u \partial_x v + \frac{\epsilon}{\delta} v \partial_y v + \frac{1}{\delta}\partial_y \zeta = 0$$



$$\mathcal{L}(t,x) = y_0 + \frac{\epsilon}{\delta}\int_0^t v(s,x,y_{bank}(s,x))ds$$

$$y = \pm\mathcal{L} \Rightarrow hv = 0$$

for banks and straight walls

$$\overline{(\cdot)} := \frac{1}{2\mathcal{L}}\int_{-\mathcal{L}(t,x)}^{\mathcal{L}(t,x)} (\cdot)(t,x,y)dy$$

Linearized problem

$$\delta(\partial_t \zeta + d\partial_x u) + \partial_y (dv) = 0$$

$$\partial_t u + \partial_x \zeta = 0$$

$$\delta \partial_t v + \partial_y \zeta = 0$$

$$d = d(y)$$

With

$$\mathcal{L} = y_0 \Rightarrow y \in [-y_0, y_0]$$

$$y = y_0 \Rightarrow dv = 0$$

for banks and straight walls

Linearized problem

$$\overline{(\cdot)} = \frac{1}{2y_0}\int_{-y_0}^{y_0} (\cdot)dy$$

$$\delta(\partial_t \zeta + d\partial_x u) + \partial_y (dv) = 0$$

$$\partial_t u + \partial_x \zeta = 0$$

$$\delta \partial_t v + \partial_y \zeta = 0$$

$$d = d(y)$$

$$y = y_0 \Rightarrow dv = 0$$

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$$d = d(y)$$

$$y = y_0 \Rightarrow dv = 0$$

$$\overline{\zeta}_{tt} - \overline{(d\zeta)}_{xx} = 0$$

This is exact

$$\zeta = \sum_{j\geq 0} \delta^j \zeta_j \,, \quad u = \sum_{j\geq 0} \delta^j u_j \,, \quad v = \sum_{j\geq 0} \delta^j v^j$$

$$\begin{aligned} \delta(\partial_t \zeta + d\partial_x u) + \partial_y (dv) &= 0 \\ \partial_t u + \partial_x \zeta &= 0 \\ \delta \partial_t v + \partial_y \zeta &= 0 \end{aligned}$$

$$\zeta = \sum_{j\geq 0} \delta^j \zeta_j\ , \quad u = \sum_{j\geq 0} \delta^j u_j\ , \quad v = \sum_{j\geq 0} \delta^j v^j$$

$$\partial_y (d v_{n+1}) = -(\partial_t \zeta_n + d\partial_x u_n) \quad \text{with BCs} \quad dv = 0$$

$$\partial_t u_{n+1} = -\partial_x \zeta_{n+1}$$

$$\partial_y \zeta_{n+1} = -\partial_t v_n \Longrightarrow \zeta_{n+1} = \overline{\zeta}_{n+1} + \boxed{Z_{n+1}} - \overline{Z}_{n+1}$$

arbitrary primitive of $-\partial_t v_n$

$$\zeta = \sum_{j\geq 0} \delta^j \zeta_j \,, \quad u = \sum_{j\geq 0} \delta^j u_j \,, \quad v = \sum_{j\geq 0} \delta^j v^j$$

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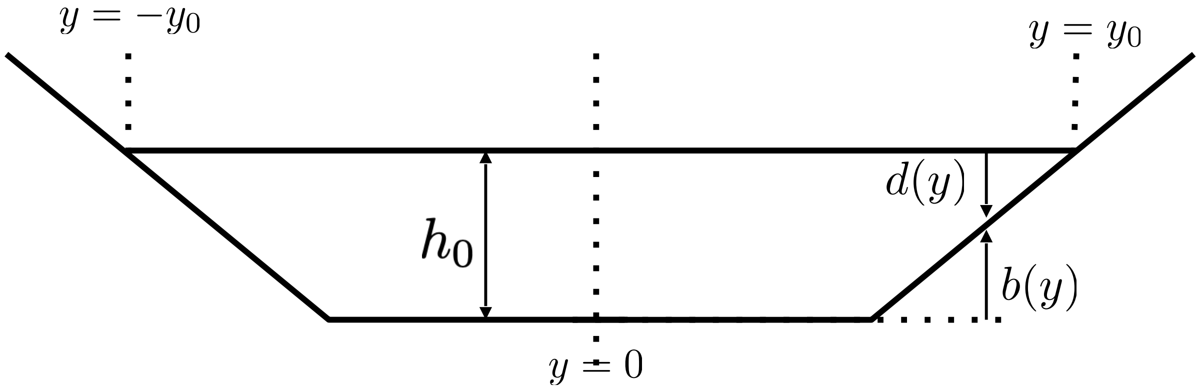
With C.I.

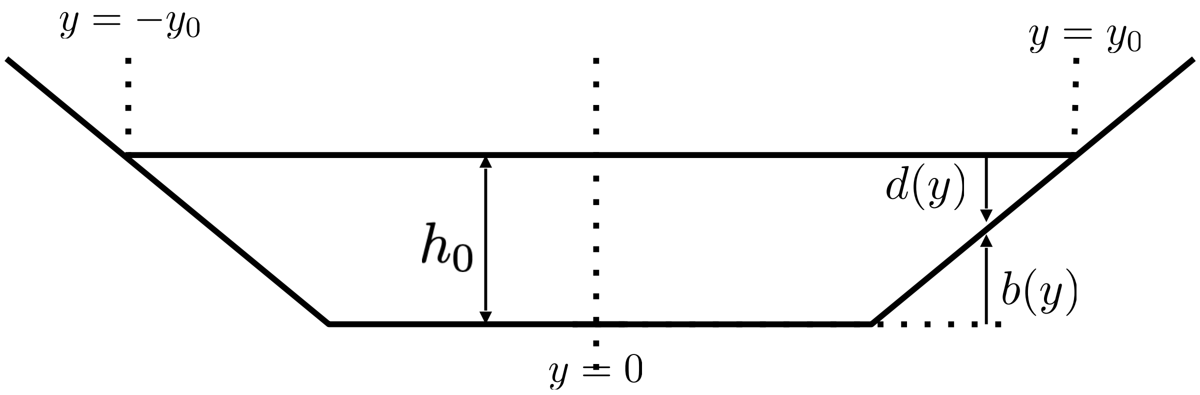
$$\zeta_0 = \bar{\zeta} \,, \quad \partial_t u_0 = -\partial_x \zeta_0 \,, \quad v_0 = 0$$



$$\zeta(x, y, t) = \overline{\zeta}(x, t) + \delta^2 (K(y) - \overline{K}) \overline{\zeta}_{xx} + \mathcal{O}(\delta^4)$$

$$K(y) := \int_{-y_0}^{y} \frac{y_0 + s - D(s)}{d(s)} ds$$

$$\mathcal{D}(y) := \int_{-y_0}^{y} d(s) ds$$



$$\zeta(x,y,t) = \overline{\zeta}(x,t) + \delta^2\left(K(y) - \overline{K}\right)\overline{\zeta}_{xx} + \mathcal{O}(\delta^4)$$

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$$\mathcal{D}(y) := \int_{-y_0}^{y} d(s)ds$$

$$\overline{\zeta}_{tt} - \overline{(d\zeta)}_{xx} = 0$$

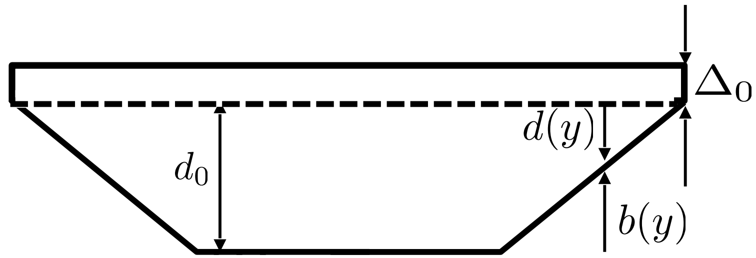
$$\overline{\zeta}_{tt} - c_0^2 \overline{\zeta}_{xx} - \boxed{\chi c_0^2 \overline{\zeta}_{xxxx}} = 0$$

$$\chi := \overline{d(y)(K(y) - \overline{K})}$$

Dispersive behaviour due to diffraction within each section

$$\boxed{\omega^2 = \kappa^2 c_0^2 (1 - \chi(\kappa y_0^2))}$$

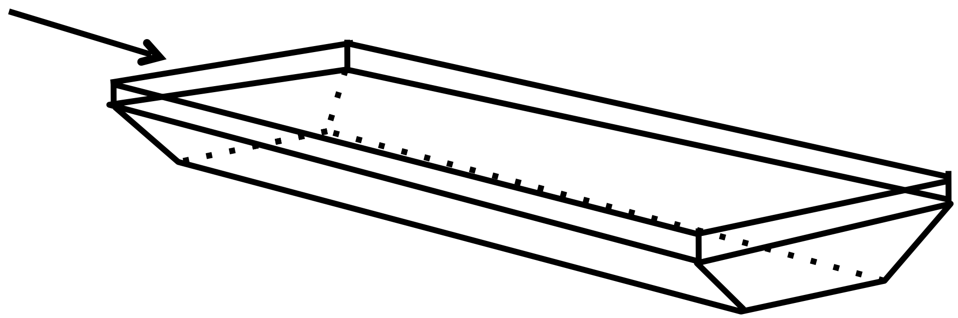
Shallow water simulations:

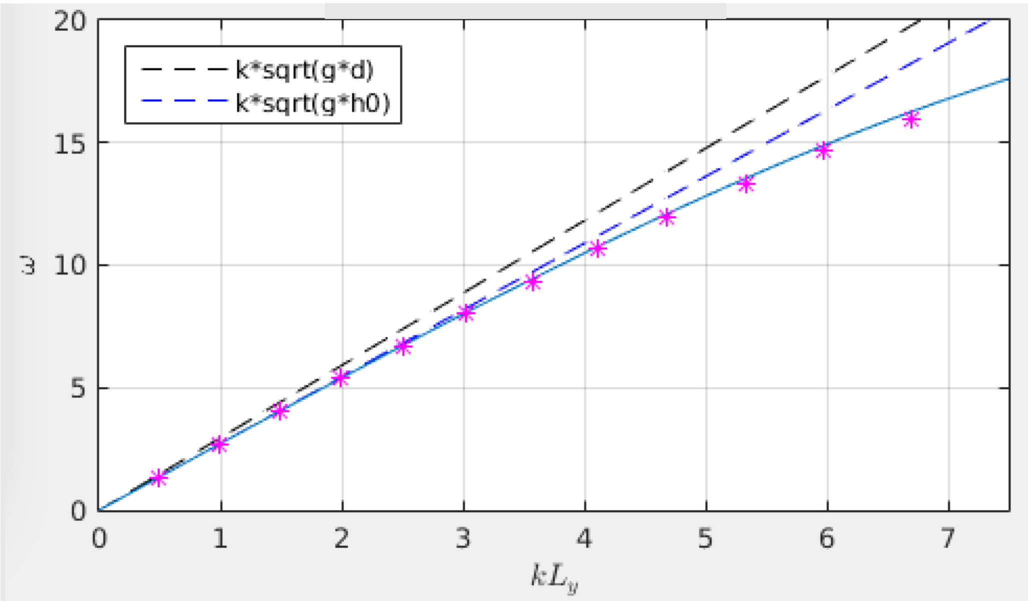


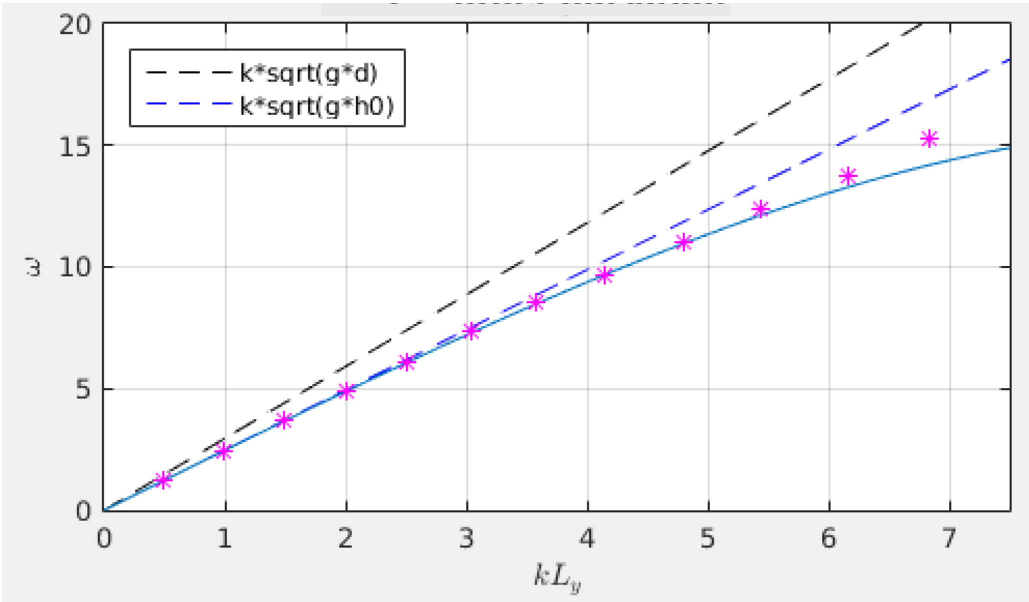
1- Linear periodic signal imposed at the inlet

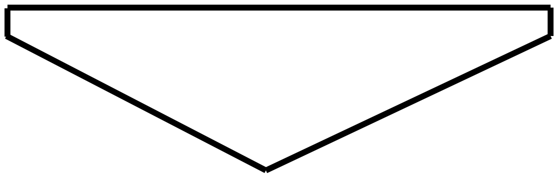
2- The whole signal is (section-)averaged

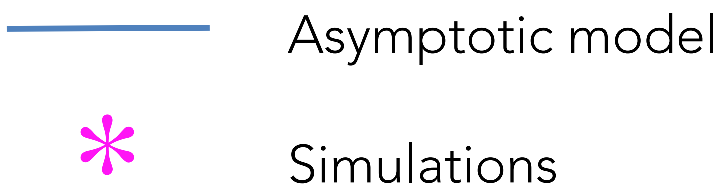
3- Period and wavelength are measured downstream

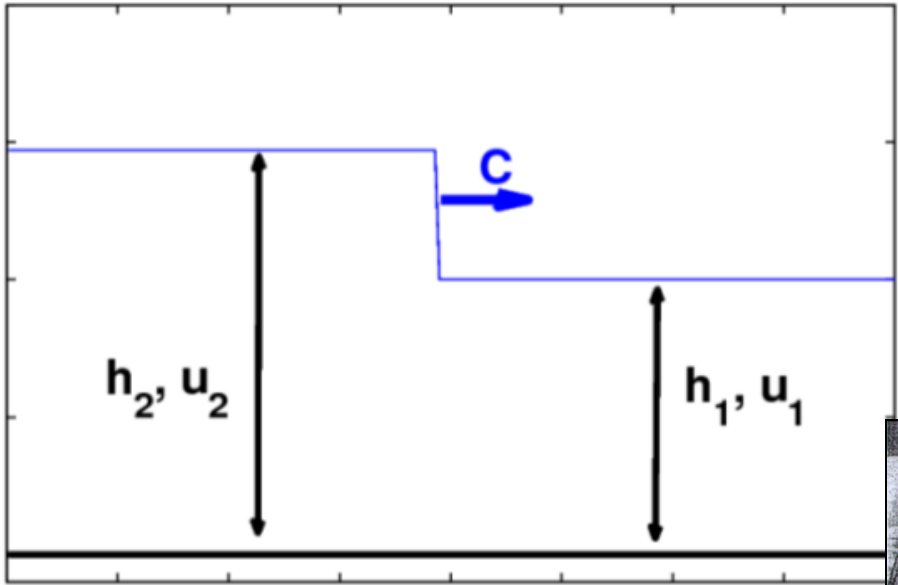


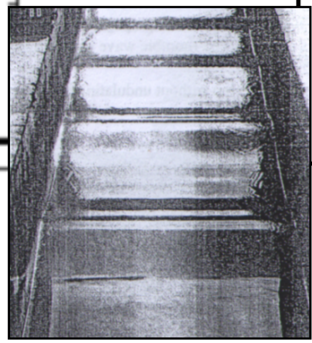

k*sqrt(g*d)
k*sqrt(g*h0)
ω
kL_y

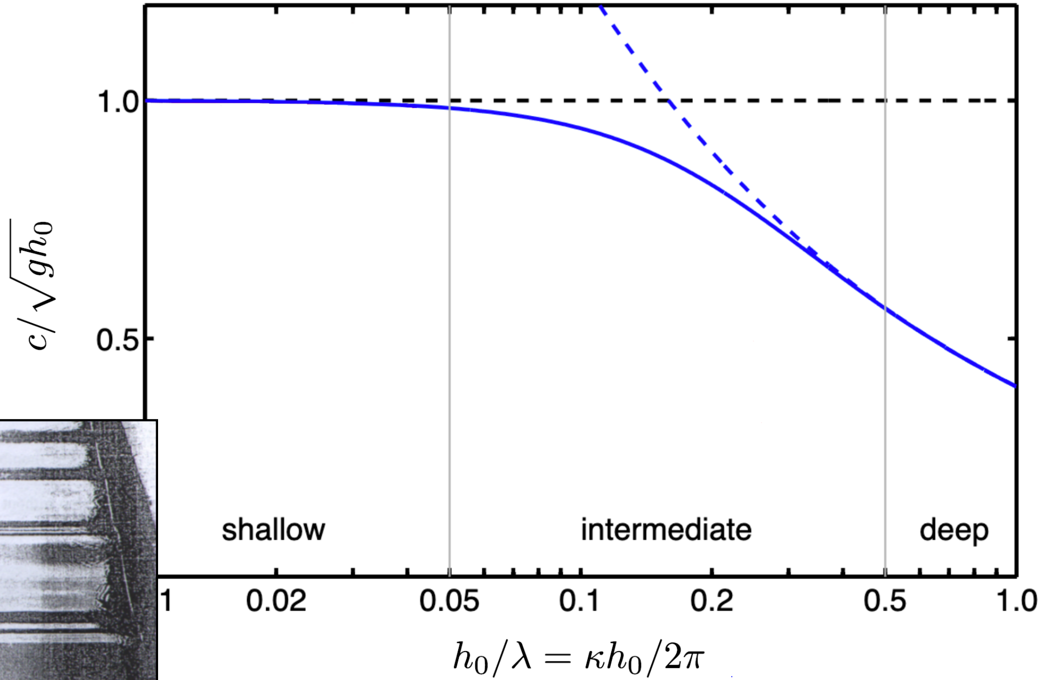

k*sqrt(g*d)
k*sqrt(g*h0)
ω
kL_y




Asymptotic model
Simulations







Bore:
given the upstream/downstream conditions and using the jump conditions:

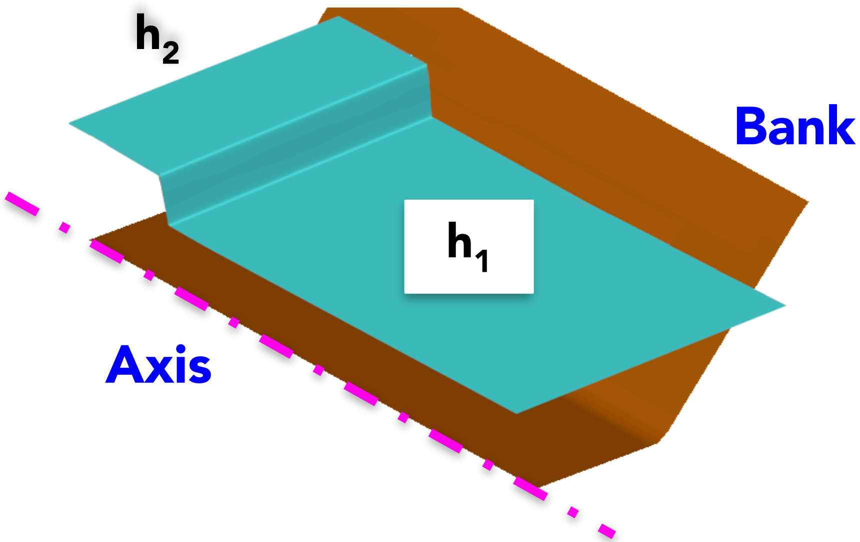
$$C = C(Fr) = C(h_2/h_1)$$

Water waves:
dispersion relation based on the linearized Euler equations (Airy theory)

$$C = C(\lambda) = \sqrt{\frac{g\lambda}{2\pi}\tanh(2\pi h/\lambda)}$$

$$C(Fr) = C(\lambda) \Longrightarrow \lambda(Fr)$$

Lemoine, La Houille Blanche, 1948



$$\overline{\zeta}_{tt} - \overline{\zeta}_{xx} - \delta^2 \chi \overline{\zeta}_{xxxx} = 0$$

$$\chi := \overline{d(y)(K(y) - \overline{K})}$$

Bore:

Jump conditions for section-averaged NLSW

Chanson, Elsevier, 2004

$$C = \boxed{C(Fr)}$$

Dispersive like-waves:

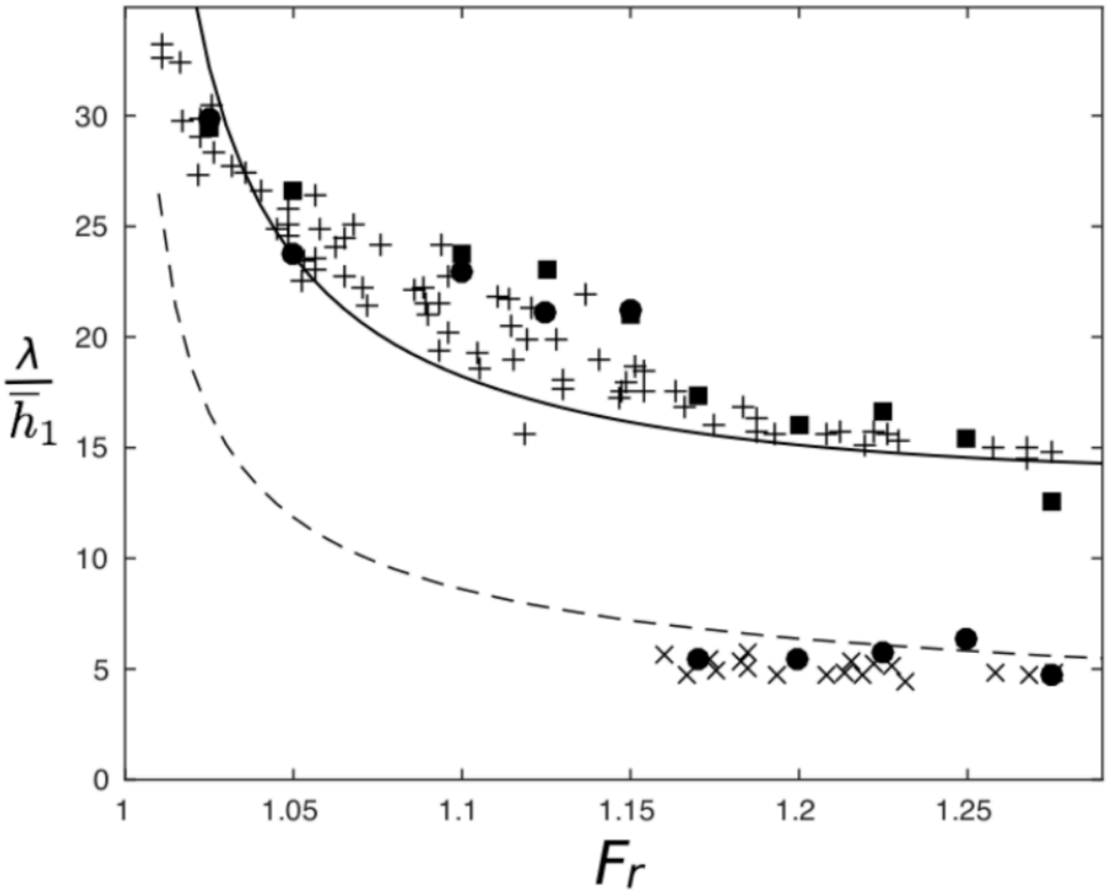
dispersion relation

$$C = \boxed{C(\lambda)}$$

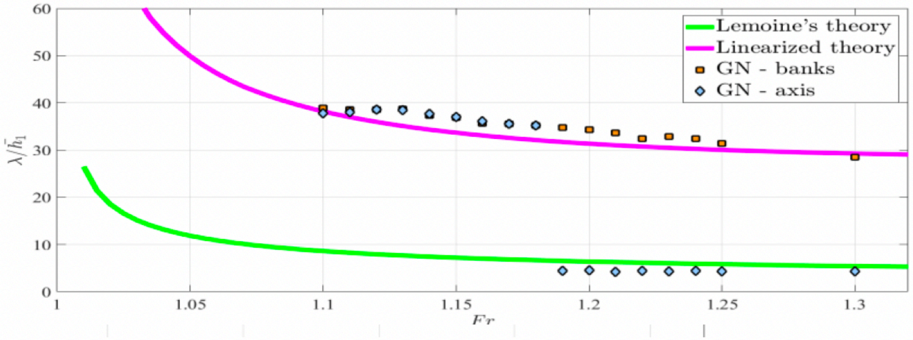
$$\boxed{\lambda = \lambda(Fr)}$$

Serre-Green-Naghdi simulations

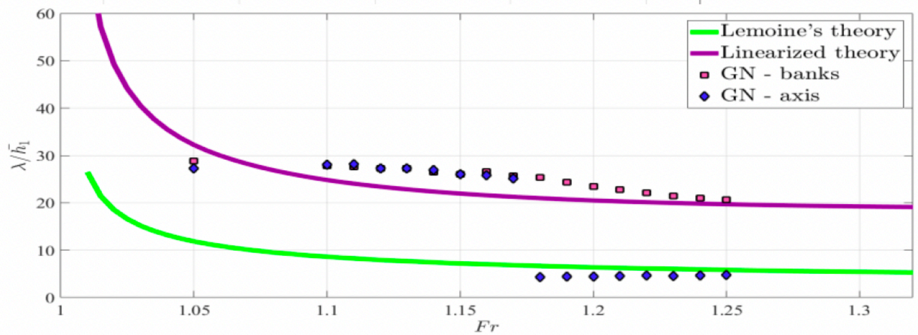
Compared to data and theory



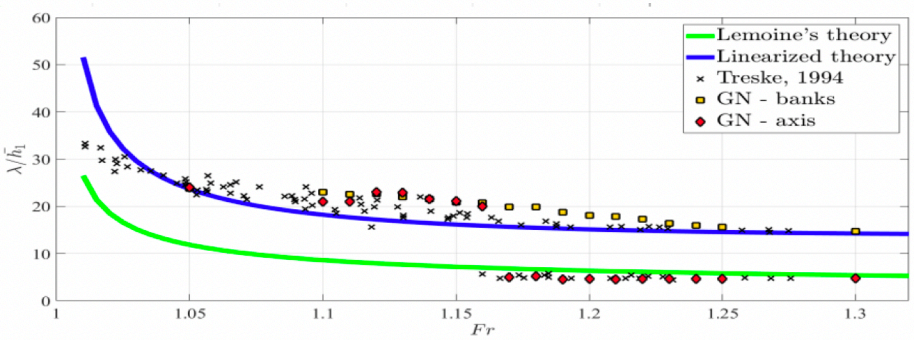
Geometrical parameters



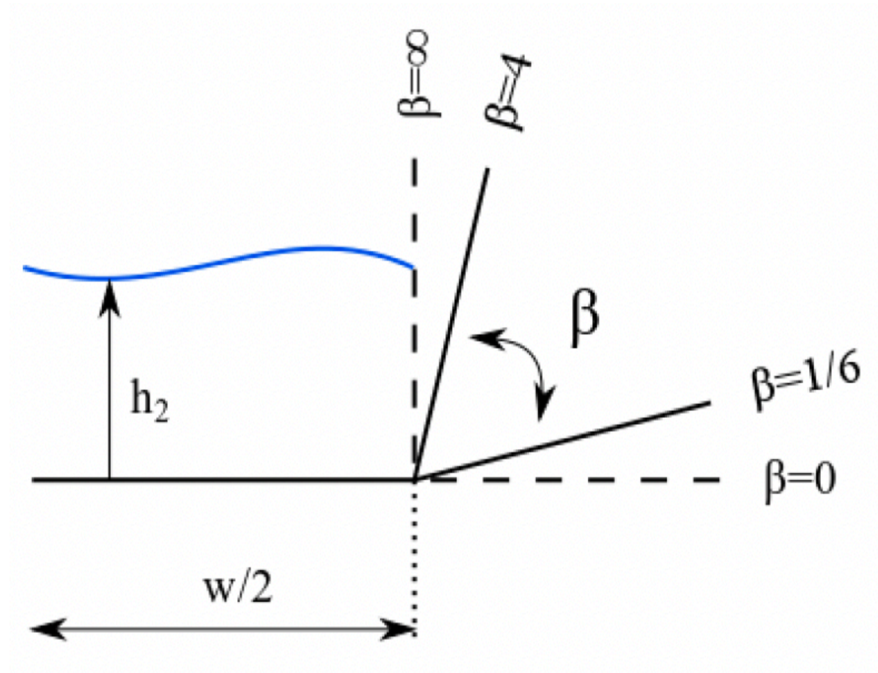
$\beta = 1/6$

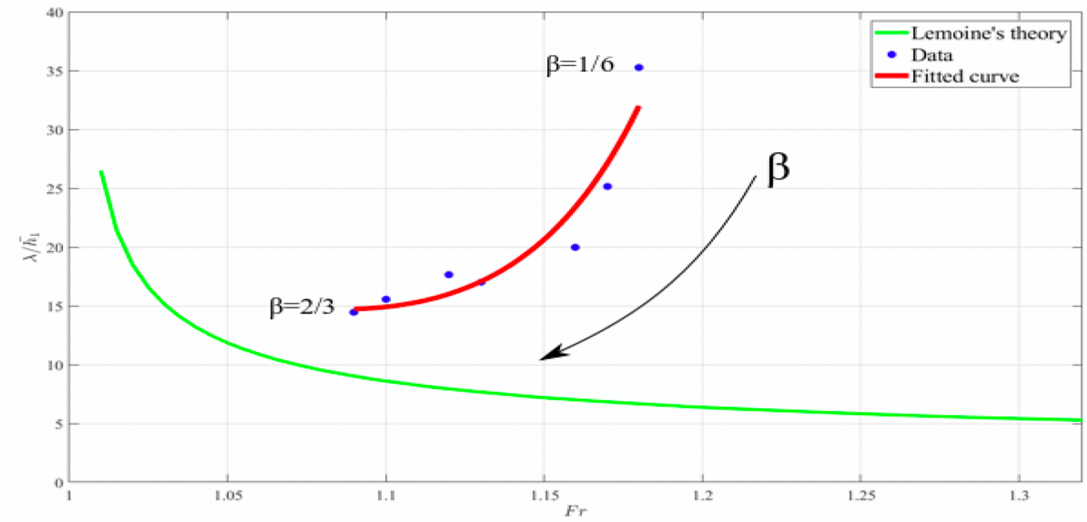


$\beta = 1/4$



$\beta = 1/3$





Ongoing foreseen work and open issues

Modelling

- Asymptotic analysis including :
  - some geometrical nonlinearity
  - "vertical" dispersion (on a Boussinesq model): connect the two dispersive behaviours
  - Nonlinear version of the above : 1D model for channels with some potential for applications (including both regimes)

- Why the transition : modulation equations (see work of El and Gavrilyuk) ?

- Coupled model with river bed morphology

Ongoing foreseen work and open issues

Numerics

- H(div) vs curl augmented approach vs current one for the elliptic pb
- Boundary conditions for the elliptic pb
- Further work on efficiency/accuracy of scheme (elliptic vs hyperbolic)
- Impact of numerical dissipation:
  - Use energy conservative fluxes (for SW)
  - Full energy conservation for GN
  - Dissipation vs solitary wave fission (cf work of El, Physics D 2016)
- Physics based dissipation

MERCI

Related work

M. Ricchiuto and A.G. Filippini, Upwind residual discretization of enhanced Boussinesq equations for wave propagation over complex bathymetries, *J.Comput.Phys. 271*, 2014

A.G. Filippini, S. Bellec, M. Colin, and M. Ricchiuto, On the nonlinear behavior of Boussinesq type models: amplitude-velocity vs amplitude-flux forms, *Coast.Eng. 99* , 2015

A.G. Filippini, M. Kazolea, and M. Ricchiuto, A flexible genuinely nonlinear approach for wave propagation, breaking and runup, *J.Comput.Phys. 310*, 2016

A.G. Filippini, M. Kazolea, and M. Ricchiuto, Hybrid finite-volume/eleent simulations of fully-nonlinear/weakly dispersive wave propagation breaking and rupnup on unstructured grids, SIAM-GS Conference, Sep 2017

A.G. Filippini, M. Kazolea, and M. Ricchiuto, A Flexible 2D Nonlinear Approach for Nonlinear Wave Propagation, Breaking and Run up, Proc.s 27th Int. Ocean and Polar Engineering Conference, 2017

A.G. Filippini, L. Arpaia, P. Bonneton, and M. Ricchiuto, Modelling analysis of tidal bore formation in convergent estuaries, Eur.J.Mech. – B/Fluids 73, 2019

R. Chassagne, A.G. Filippini, P. Bonneton, and M. Ricchiuto, Dispersive and dispersive-like bores in channels with sloping banks, *Journal of Fluid Mechanics* 870, pp. 595-616, 2019

M. Kazolea, A.G. Filippini, M. Ricchiuto, Low dispersion finite volume/element discretization of the enhanced Green–Naghdi equations for wave propagation, breaking and runup on unstructured meshes, *Ocean Mod.* 182, 2023

M. Kazolea, and M. Ricchiuto, M. Kazolea and M. Ricchiuto, Full nonlinearity in weakly dispersive Boussinesq models: luxury or necessity ?, *J. Hydraul. Eng,* to appear