Turnpike property of optimal control problems with symmetries

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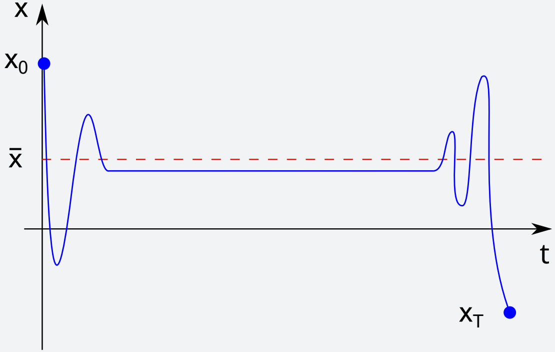
Outline

Outline

Turnpike property in optimal control

Quasi-static behavior of solutions of optimal control problem in large time for different initial and final conditions.



Turnpike appears in different classes of OCPs:

- Finite dimensional (ODE) [Trélat-Zuazua 2015, Faulwasser et al. 2015]
- Infinite dimensional (PDE) [Grüne et al. 2019]
- Discrete systems (autonomous and non autonomous) [Grüne et al. 2018]

Turnpike appears in economical, biological, physical systems

Applications of turnpike:

- Asymptotic analysis [Grüne et al. 2018, Trélat-Zuazua 2015]
- Numerical methods [Trélat-Zuazua 2015, Cots et al. 2021]
- Error analysis in MPC [Grüne et al. 2019]
- Sub-optimal control strategies [Caillau et al. 2022]

Partial turnpike

Symmetry induced turnpike in mechanical systems [Faulwasser et al. 2022]

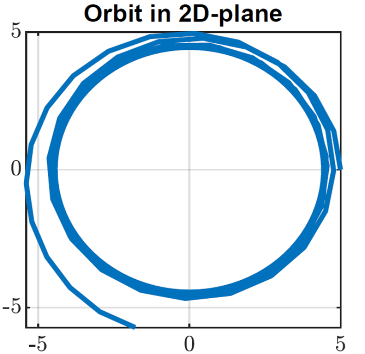


Figure 1: Turnpike towards rotational orbit in 2D Kepler [Faulwasser et al. 2022]

Outline

Optimal control problem (OCP)

Find control $u(t) \in \mathbb{R}^m$ and trajectory $x(t) \in M$ solution of

$$\begin{aligned} \min_{u} \quad & J(u) = \int_0^T f^0(x(t), u(t))dt \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t)) \\ & x(0) = 0 \end{aligned}$$

Hamiltonian (Pontryagin Min Principle) $H(x, \lambda, u) = \langle \lambda, f(x, u) \rangle + f^0(x, u)$

Static problem

$$\begin{aligned} \min_{\bar{u} \in \mathbb{R}^m, \bar{x} \in M} \quad & f^0(\bar{x}, \bar{u}) \\ & f(\bar{x}, \bar{u}) = 0, \end{aligned}$$

Lagrangian (Karush–Kuhn–Tucker) $L(\bar{x}, \bar{\lambda}, \bar{u}) = \langle \bar{\lambda}, f(\bar{x}, \bar{u}) \rangle + f^0(\bar{x}, \bar{u})$
with $\bar{\lambda}$ Lagrange multiplier

Classical turnpike property

Let $(\bar{x}, \bar{u})$ be a solution of the static OCP and $\bar{\lambda}$ the Lagrange multiplier

Local exponential turpike property [Trélat and Zuazua 2015]

There exist positive constants ε, μ, C and T_0 such that

$$\text{if } \|x(0) - \bar{x}\| + \|\bar{\lambda}\| \leq \varepsilon$$

then for any $T > T_0$ there holds

$$\|x(t) - \bar{x}\| + \|u(t) - \bar{u}\| \leq C\left(e^{-\mu t} + e^{-\mu(T-t)}\right), \qquad t \in [0, T]$$

Classical turnpike theorem [Trélat and Zuazua 2015]

Under conditions on Hamiltonian H (hyperbolicity), local exp. turnpike holds

Outline

Lie group action

Let M be a smooth manifold

Lie group action

Lie group G acts on M by smooth action $\Phi : G \times M \to M$ satisfying

- $\Phi(e, x) = x$ for all $x \in M$
- $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ for all $x \in M$ and $g, h \in G$

For any $g \in G$, let $\Phi_g : x \mapsto \Phi(g, x) = \Phi_g(x) = g \cdot x$

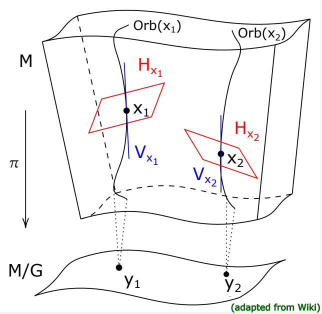
- Orbit of $x \in M$ is $\mathrm{Orb}(x) = \{\Phi_g(x) \ : \ g \in G\}$
- Let $\xi \in \mathfrak{g}$, its flow from $x \in M$ is defined by $\phi^\xi(t) = \Phi(\exp(\xi t), x)$

Trim

x solution of $\dot{x} = f(x, u)$ with const u and s.t. $x(t) = \phi^\xi(t)$ for some $\xi \in \mathfrak{g}$

Principal bundle

- G is free and proper $\Rightarrow$ principal bundle $\pi : M \to M/G$ = space of orbits
- principal connection $\mathcal{A} : TM \to \mathfrak{g}$ is $\mathfrak{g}$-valued one form on M s.t. $\mathcal{A}(gv) = \mathrm{Ad}_g \mathcal{A}(v)$ and $\mathcal{A}(\frac{d}{dt}\phi^\xi(t)\big|_{t=0}) = \xi$
- splitting $T_x M = T_x \mathrm{Orb}(x) \oplus \ker(\mathcal{A}_x) = V_x \oplus H_x$



Example: trivial bundle

principal bundle $\pi : P \times G \to P$ with Maurer-Cartan form $\mathcal{A}_{(p,g)} = g^{-1} \circ dg$

Symmetry in optimal control

Optimal control problem (OCP)

Find control $u(t) \in \mathbb{R}^m$ and trajectory $x(t) \in M$ solution of

$$\begin{aligned} \min_u \quad & J(u) = \int_0^T f^0(x(t), u(t))dt \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t)) \\ & x(0) = x_0. \end{aligned}$$

Lie group G acts on M by $\Phi_g : M \to M$ and $d\Phi_g : TM \to TM$ for any $g \in G$

OCP with G symmetry

- f^0 invariant w.r.t G : $f^0(\Phi_g(x), u) = f^0(x, u)$
- f equivariant w.r.t G : $f(\Phi_g(x), u) = d_x\Phi_g f(x, u)$

Reduction

Principal connection $\mathcal{A} : TM \to \mathfrak{g}$ defines $TM/G \overset{\mathcal{A}}{\simeq} \underbrace{T(M/G) \oplus (M \times \mathfrak{g})/G}_{\text{bundle over } M/G}$

Reduced OCP [Ohsawa 2013]

Find control $u(t) \in \mathbb{R}^m$ and trajectory $y(t) \in M/G,\ \xi(t) \in \tilde{\mathfrak{g}} = (M \times \mathfrak{g})/G$ solution of

$$\begin{aligned} \min_u \quad & J(u) = \int_0^T f^0_{M/G}(y(t), u(t)) dt \\ \text{s.t.} \quad & \dot{y}(t) = f_{M/G}(y(t), u(t)) \\ & \xi(t) = f_{\tilde{\mathfrak{g}}}(y(t), u(t)) \\ & y(0) = \pi(x_0). \end{aligned}$$

- $f_{M/G}(y, u) = d\pi(x) \circ f(x, u)$
- $\xi(t) = f_{\tilde{\mathfrak{g}}}(y, u)\ :\ [x, \mathcal{A}_x(\dot{x})]_G = [x, \mathcal{A}_x(f(x, u))]_G$ with $[\cdot, \cdot]_G$ equiv. class

In (y, g)-coordinates : $\xi(t) = (y, \zeta(t) + \mathcal{A}_{(y,e)}(\dot{y}, 0)),\ \zeta(t) = g^{-1}\dot{g} \in \mathfrak{g}$

Example: 2D Kepler problem = 2 body problem in 2-dim plane

State $(r, v_r, \theta, v_\theta) \in \mathbb{R}^2 \setminus \{0\} \times TS^1$ and $G = S^1$ acts on θ by translations

$$\min_u J(u) = \int_0^T f^0(r(t), v_r(t), v_\theta(t), u(t))dt$$

$$\begin{cases} \dot{r} = v_r \\ \dot{v}_r = r v_\theta^2 - \frac{1}{r^2} + u_1 \\ \dot{\theta} = v_\theta \\ \dot{v}_\theta = -2\frac{v_r v_\theta}{r} + \frac{u_2}{r^2} \end{cases}$$

- Trivial bundle $\mathbb{R}^2 \setminus \{0\} \times TS^1 \simeq \big(\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}\big) \times S^1$
- Maurer-Cartan connection $\mathcal{A} : (\theta, v_\theta) \mapsto v_\theta$

$$\min_u J(u) = \int_0^T f^0(r(t), v_r(t), v_\theta(t), u(t))dt$$

Reduced problem:

$$\begin{cases} \dot{r} = v_r \\ \dot{v}_r = r v_\theta^2 - \frac{1}{r^2} + u_1 \\ \dot{v}_\theta = -2\frac{v_r v_\theta}{r} + \frac{u_2}{r^2} \end{cases} \quad \text{and} \quad \xi(t) = v_\theta(t)$$

OCP with symmetries

Consider OCP invariant w.r.t. G-action on M

$$\begin{aligned} \min_u \quad & J(u) = \int_0^T f^0(x(t), u(t))dt \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t)) \\ & x(0) = x_0. \end{aligned}$$

Static OCP = static reduced OCP

$\bar{u} \in \mathbb{R}^m$ and $\bar{y} \in M/G,\ \xi \in \tilde{\mathfrak{g}} = (M \times \mathfrak{g})/G$ solution of

$$\begin{aligned} & \min_{\bar{u}} f^0_{M/G}(\bar{y}, \bar{u}) \\ & 0 = f_{M/G}(\bar{y}, \bar{u}) \\ & \xi = f_{\tilde{\mathfrak{g}}}(\bar{y}, \bar{u}) \end{aligned}$$

$\xi = f_{\tilde{\mathfrak{g}}}(\bar{y}, \bar{u})$ leads to $\dot{g}(t) = g(t)\zeta(\bar{y}, \bar{u}),\ \zeta(\bar{y}, \bar{u}) \in \mathfrak{g}$

Static solution = trim corresponding to $\bar{u}$ and $\zeta(\bar{y}, \bar{u}) \in \mathfrak{g}$

Turnpike OCP with symmetries

- Trivial bundle case: $M = M/G \times G$: global coordinates $x = (y, g)$
- General case: $M = M/G \times G$ with $x = (y, g)$ locally near fixed $y_0 \in M/G$

In coordinates (y, g): $\bar{y}$ can be lifted to $\bar{x}_0 = (\bar{y}, g_0)$ and the trim is defined by

$$\bar{x}(t) = \phi^{\zeta(\bar{y},\bar{u})t}(\bar{y}, g_0)$$

Theorem 1

If the reduced problem admits a classical local exp. turnpike
$\Rightarrow$ full OCP admits the local exp. turnpike toward trim $\bar{x}(t) = \phi^{\zeta(\bar{y},\bar{u})t}(\bar{y}, g_0)$:
There exist positive constants ε, μ, C and T_0 such that

$$\text{if } \|\pi(x(0)) - \bar{y}\| + \|\bar{\lambda}_y\| \leq \varepsilon$$

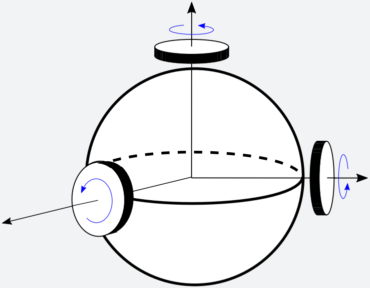
then for any $T > T_0$ there holds

$$\|x(t) - \bar{x}(t)\| + \|u(t) - \bar{u}\| \leq C\left(e^{-\mu t} + e^{-\mu(T-t)}\right), \qquad t \in [0, T]$$

Outline

Model of rigid body with rotors

Configuration $Q = SO(3) \times S^1 \times S^1 \times S^1$ with $R \in SO(3)$ and $\theta \in S^1 \times S^1 \times S^1$



$G = SO(3)$ acts on Q by rotations of R

The motions are described by Lagrange–d'Alembert principle and Lagrangian

$$L(R, \dot{R}, \theta, \dot{\theta}) = \frac{1}{2}\langle R^{-1}\dot{R}, IR^{-1}\dot{R}\rangle + \frac{1}{2}\langle R^{-1}\dot{R} + \dot{\theta}, K\left(R^{-1}\dot{R} + \dot{\theta}\right)\rangle$$

I, K inertia tensors : I of the rigid body and K of rotors

Model of rigid body with rotors

$$L(R,\dot{R},\theta,\dot{\theta}) = \frac{1}{2}\langle R^{-1}\dot{R}, IR^{-1}\dot{R}\rangle + \frac{1}{2}\langle R^{-1}\dot{R}+\dot{\theta}, K\left(R^{-1}\dot{R}+\dot{\theta}\right)\rangle$$

Control system = controlled Euler-Lagrange equations

State $(R, v_R, \theta, v_\theta) \in M = TSO(3) \times T(S^1 \times S^1 \times S^1)$, control $u \in \mathbb{R}^3$

$$\begin{aligned} \frac{d}{dt}\frac{\partial}{\partial \dot{R}}L - \frac{\partial}{\partial R}L = 0 \\ \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}L - \frac{\partial}{\partial \theta}L = u \end{aligned} \quad \Rightarrow \quad \begin{cases} \dot{R} = v_R \\ \dot{\theta} = v_\theta \\ \dot{v}_R = v_R\Omega + Rf_{v_R}(\Omega, v_\theta, u) \\ \dot{v}_\theta = f_\theta(\Omega, v_\theta, u) \end{cases}$$

we denote $\Omega = R^{-1}\dot{R} = R^{-1}v_R$

L is invariant w.r.t. $G \Rightarrow$ controlled E-L eq. equivariant w.r.t. G action

OCP invariant w.r.t $G = SO(3)$

State $x = (R, v_R, \theta, v_\theta) \in M = TSO(3) \times T(S^1 \times S^1 \times S^1)$, control $u \in \mathbb{R}^3$

$$\min_u J(u) = \int_0^T f^0(\Omega, v_\theta, \theta, u)dt$$
$$\dot{x} = f(x, u)$$
$$x(0) = x_0$$

with $\Omega = R^{-1}\dot{v}_R$

$TSO(3) \times T(S^1 \times S^1 \times S^1) \simeq \big(\mathfrak{g} \times T(S^1 \times S^1 \times S^1)\big) \times SO(3)$ trivial bundle

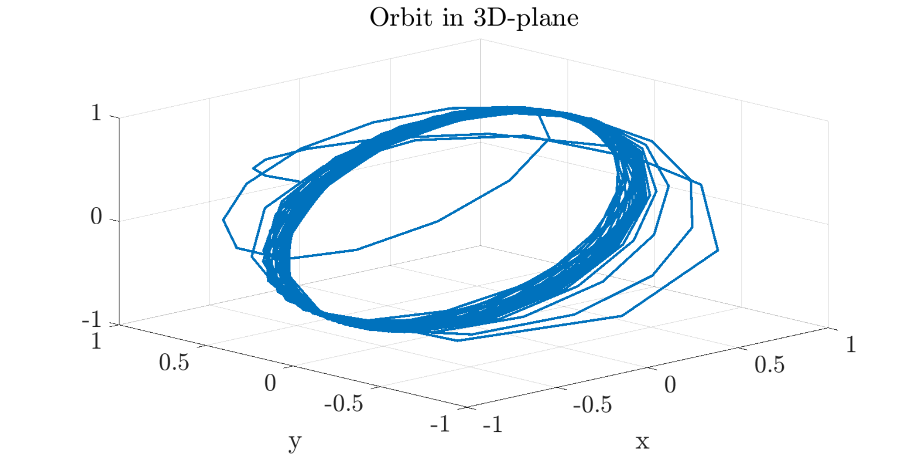
Static reduced OCP

State $(\bar{\Omega} = R^{-1}v_R, \bar{\theta}, \bar{v}_\theta) \in M/G = \mathfrak{g} \times T(S^1 \times S^1 \times S^1)$

$$\min_{u \in \mathbb{R}^3} f^0(\bar{\Omega}, \bar{v}_\theta, \bar{\theta}, \bar{u})$$
$$0 = f_{M/G}(\bar{\Omega}, \bar{v}_\theta, \bar{u}) \quad \text{and} \quad \xi(t) = \bar{\Omega}$$

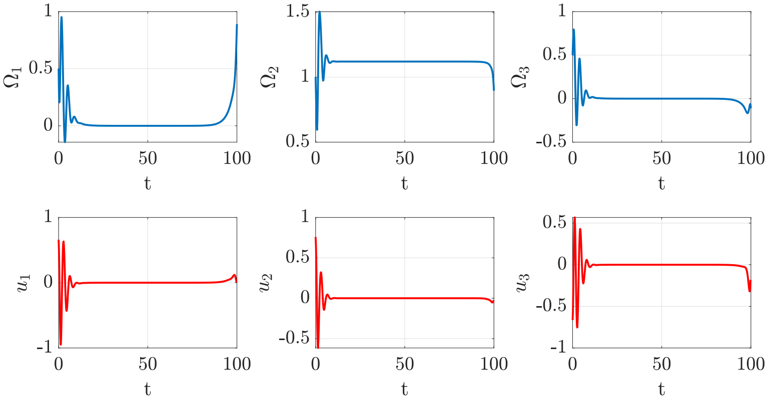
Trim turnpike: orbit $R(t) = R_0 \exp(\bar{\Omega}t)$

Numerical results



Cost $f^0 = (\Omega - e)^2 + \theta^2 + u^2$ with $e = (1, 1, 1)$

Numerical results



Outline

Conclusions and outlook

Conclusions:

- Framework for turnpike in OCP with symmetries
- Local exponential turnpike of OCP with symmetries and free final state
- Theoretical results validated on example from mechanics

Outlook:

- Global turnpike
- Fixed final conditions
- Group action on both state and control

Thank you for your attention!

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