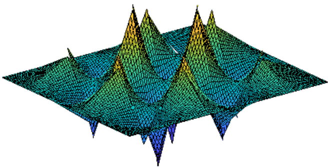
Modern Coarse Spaces in Domain Decomposition

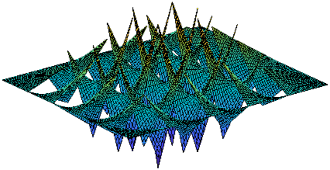
Eigenmodes of the iteration map for Schwarz methods with crosspoints

Laurence Halpern and Martin Gander

LAGA-UNIVERSITÉ SORBONNE PARIS-NORD

SMAI 2023, Le Gosier, Gaudeloupe





Joint work with M. J. Gander (Genève) and F. Cuvelier (LAGA UP13).
Collaboration with K. Santugini.

Frame of the analysis

- Study the iteration map

Frame of the analysis

- Study the iteration map
- Find its eigenmodes

Frame of the analysis

- Study the iteration map
- Find its eigenmodes
- Define from them the optimal coarse space

Frame of the analysis

- Study the iteration map
- Find its eigenmodes
- Define from them the optimal coarse space
- Design an optimized coarse space

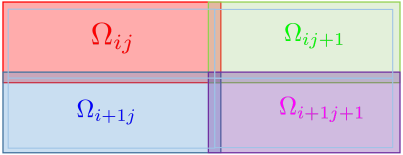
The modal analysis with crosspoints

The subdomain decomposition

Ω_{11}	$\ldots$	$\ldots$	$\ldots$	$\ldots$	Ω_{1N}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\vdots$	$\vdots$	Ω_{ij}	Ω_{ij+1}	$\vdots$	$\vdots$
$\vdots$	$\vdots$	Ω_{i+1j}	Ω_{i+1j+1}	$\vdots$	$\vdots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\vdots$	$\ldots$
Ω_{M1}	$\ldots$	$\ldots$	$\ldots$	$\vdots$	Ω_{MN}

The subdomain decomposition

Ω_{11}	$\ldots$	$\ldots$	$\ldots$	$\ldots$	Ω_{1N}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\vdots$	$\vdots$	Ω_{ij}	Ω_{ij+1}	$\vdots$	$\vdots$
$\vdots$	$\vdots$	Ω_{i+1j}	Ω_{i+1j+1}	$\vdots$	$\vdots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\vdots$	$\ldots$
Ω_{M1}	$\ldots$	$\ldots$	$\ldots$	$\vdots$	Ω_{MN}



Overlap size $2L$.

The subdomain solution

$$-\Delta u_{ij} = 0$$

Ansatz for the solution by separation of variables

$$u_{ij} = (a_{ij} \sin \zeta(x-x_{j-1}) + a'_{ij} \sin \zeta(x-x_j))(b_{ij} \sinh \zeta(y-y_{i-1}) + b'_{ij} \sinh \zeta(y-y_i))$$

a_{11} b_{11}		a_{1j}, a'_{1j} b_{1j}	a_{1j+1}, a'_{1j+1} b_{1j+1}		a'_{1N} b_{1N}
a_{21} b_{21}, b'_{21}					
a_{i1} b_{i1}, b'_{i1}		a_{ij}, a'_{ij} b_{ij}, b'_{ij}	a_{ij+1}, a'_{ij+1} b_{ij+1}, b'_{ij+1}		a'_{iN} b_{iN}, b'_{iN}
a_{i+11} b_{i+11}, b'_{i+11}		a_{i+1j}, a'_{i+1j} b_{i+1j}, b'_{i+1j}	a_{i+1j+1}, a'_{i+1j+1} b_{i+1j+1}, b'_{i+1j+1}		a'_{i+1N} b_{i+1N}, b'_{i+1N}
a_{M1} b'_{M1}		a_{Mj}, a'_{Mj} b'_{Mj}	a_{Mj+1}, a'_{Mj+1} b'_{Mj+1}		a'_{MN} b'_{MN}

The subdomain solution

$$-\Delta u_{ij} = 0$$

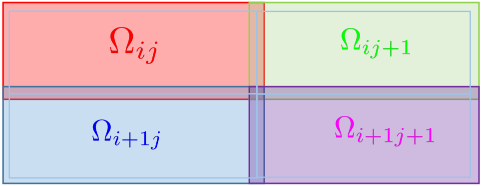
Ansatz for the solution by separation of variables

$$u_{ij} = (a_{ij}\sin\zeta(x-x_{j-1})+a'_{ij}\sin\zeta(x-x_j))(b_{ij}\sinh\zeta(y-y_{i-1})+b'_{ij}\sinh\zeta(y-y_i))$$

a_{11} b_{11}		a_{1j}, a'_{1j} b_{1j}	a_{1j+1}, a'_{1j+1} b_{1j+1}		a'_{1N} b_{1N}
a_{21} b_{21}, b'_{21}					
a_{i1} b_{i1}, b'_{i1}		a_{ij}, a'_{ij} b_{ij}, b'_{ij}	a_{ij+1}, a'_{ij+1} b_{ij+1}, b'_{ij+1}		a'_{iN} b_{iN}, b'_{iN}
a_{i+11} b_{i+11}, b'_{i+11}		a_{i+1j}, a'_{i+1j} b_{i+1j}, b'_{i+1j}	a_{i+1j+1}, a'_{i+1j+1} b_{i+1j+1}, b'_{i+1j+1}		a'_{i+1N} b_{i+1N}, b'_{i+1N}
a_{M1} b'_{M1}		a_{Mj}, a'_{Mj} b'_{Mj}	a_{Mj+1}, a'_{Mj+1} b'_{Mj+1}		a'_{MN} b'_{MN}

$$u_{ij} = (a_{ij}\sinh\zeta(x-x_{j-1})+a'_{ij}\sinh\zeta(x-x_j))(b_{ij}\sin\zeta(y-y_{i-1})+b'_{ij}\sin\zeta(y-y_i))$$

The iteration map



$u = \{u_{ij}\}$ harmonic in the subdomains $\to$ $v = \{v_{ij}\}$ harmonic in the subdomains, coupled as a slight generalization with transmission condition,

$$\begin{aligned} \partial_x v_{ij} + p v_{ij} &= \partial_x u_{ij+1} + p u_{ij+1}, \quad x = x_j + L, \\ -\partial_x v_{ij+1} + p v_{ij+1} &= -\partial_x u_{ij} + p u_{ij}, \quad x = x_j - L. \end{aligned}$$

- $p = +\infty$, $L > 0$, parallel Schwarz
- $0 < p < +\infty$, $L > 0$, parallel overlapping Robin-Schwarz
- $0 < p < +\infty$, $L = 0$, parallel nonoverlapping Robin-Schwarz.

Eigenmodes of the iteration map

(λ, u) such that $v = \lambda u$.

$$u_{ij} = (a_{ij} \sin\zeta(x - x_{j-1}) + a'_{ij} \sin\zeta(x - x_j))(b_{ij} \sinh\zeta(y - y_{i-1}) + b'_{ij} \sinh\zeta(y - y_i))$$

From now on, all subdomains are squares with length H.

Eigenmodes of the iteration map

(λ, u) such that $v = \lambda u$.

$$u_{ij} = (a_{ij} \sin\zeta(x - x_{j-1}) + a'_{ij} \sin\zeta(x - x_j))(b_{ij} \sinh\zeta(y - y_{i-1}) + b'_{ij} \sinh\zeta(y - y_i))$$

From now on, all subdomains are squares with length H.

Theorem, 2020

$$\lambda(Z^+ + \delta_x^{(j)} Z^0) a_{ij} b_{ij} = (Z^0 + \delta_x^{(j+1)} Z^-) a_{ij+1} b_{ij+1}$$
$$\lambda(Z^0 + \delta_x^{(j+1)} Z^+) a_{ij+1} b_{ij+1} = (Z^- + \delta_x^{(j)} Z^0) a_{ij} b_{ij}$$

$$\lambda(Z_h^+ + \delta_y^{(i)} Z_h^0) a_{ij} b_{ij} = (Z_h^0 + \delta_y^{(i+1)} Z_h^-) a_{i+1j} b_{i+1j},$$
$$\lambda(Z_h^0 + \delta_y^{(i+1)} Z_h^+) a_{i+1j} b_{i+1j} = (Z_h^- + \delta_y^{(i)} Z_h^0) a_{ij} b_{ij}$$

$$Z^- = \zeta \cos\zeta(H - L) - p \sin\zeta(H - L), \quad Z_h^- = \zeta \cosh\zeta(H - L) - p \sinh\zeta(H - L),$$
$$Z^+ = \zeta \cos\zeta(H + L) + p \sin\zeta(H + L), \quad Z_h^+ = \zeta \cosh\zeta(H + L) + p \sinh\zeta(H + L)$$
$$Z^0 = \zeta \cos\zeta L + p \sin\zeta L, \quad Z_h^0 = \zeta \cosh\zeta L + p \sinh\zeta L$$

$$j = 2, \cdots N - 1, \ \frac{a'_{ij}}{a_{ij}} = \frac{a'_{1j}}{a_{1j}} := \delta_x^{(j)}, \qquad i = 2, \cdots M - 1, \frac{b'_{ij}}{b_{ij}} = \frac{b'_{i1}}{b_{i1}} := \delta_y^{(i)}.$$

Dispersion relation (equation for modes) and eigenvalues

$$
\begin{aligned}
Z^- &= \zeta \cos\zeta(H-L) - p\sin\zeta(H-L), & Z_h^- &= \zeta \cosh\zeta(H-L) - p\sinh\zeta(H-L),\\
Z^+ &= \zeta \cos\zeta(H+L) + p\sin\zeta(H+L), & Z_h^+ &= \zeta \cosh\zeta(H+L) + p\sinh\zeta(H+L)\\
Z^0 &= \zeta \cos\zeta L + p\sin\zeta L, & Z_h^0 &= \zeta \cosh\zeta L + p\sinh\zeta L
\end{aligned}
$$

Theorem

$$\delta_x^{(j)} = \frac{a'_{ij}}{a_{ij}}, \qquad \lambda^2 = \frac{Z^- + \delta_x^{(j)} Z^0}{Z^+ + \delta_x^{(j)} Z^0}\,\frac{Z^0 + \delta_x^{(j+1)} Z^-}{Z^0 + \delta_x^{(j+1)} Z^+}, \quad j = 2\cdots N-1,$$

Similarly in y

$$\delta_y^{(i)} = \frac{b'_{ij}}{b_{ij}}, \qquad \lambda^2 = \frac{Z_h^- + \delta_y^{(i)} Z_h^0}{Z_h^+ + \delta_y^{(i)} Z_h^0}\,\frac{Z_h^0 + \delta_y^{(i+1)} Z_h^-}{Z_h^0 + \delta_y^{(i+1)} Z_h^+}, \quad i = 2\cdots M-1.$$

$\zeta = 0$: affine modes.

The affine modes

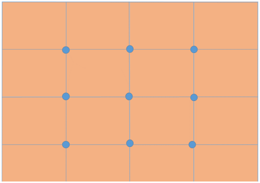
Theorem

1. $2(N-1)$ affine modes for $N \times N$ subdomains,
2. No affine modes for $M \neq N$.

The affine modes

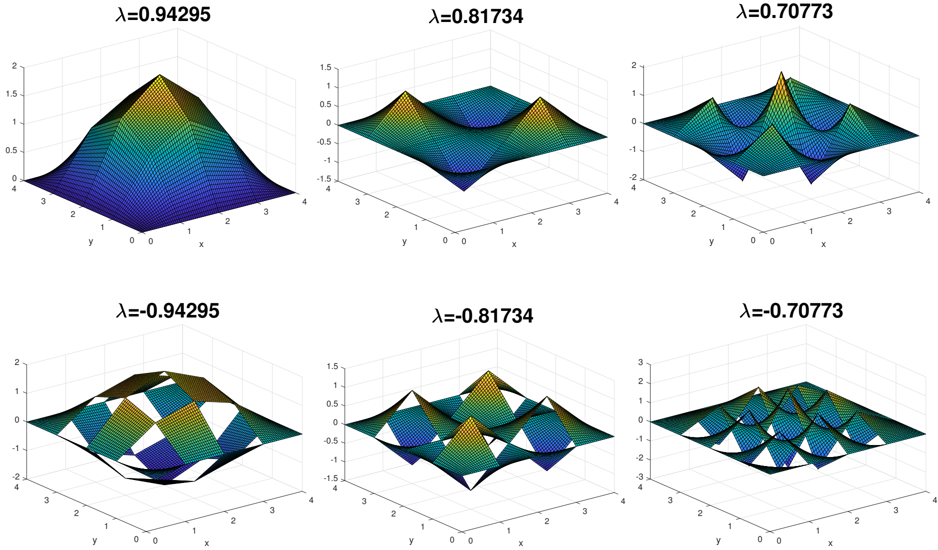
Theorem

1. $2(N-1)$ affine modes for $N \times N$ subdomains,
2. No affine modes for $M \neq N$.



Example : 4 × 4 subdomains

**



2×2 SUBDOMAINS
BASIC ELEMENTS

Dispersion relation (equation for modes) and eigenvalues

$$
\begin{aligned}
Z^- &= \zeta\cos\zeta(H-L) - p\sin\zeta(H-L), & Z_h^- &= \zeta\cosh\zeta(H-L) - p\sinh\zeta(H-L),\\
Z^+ &= \zeta\cos\zeta(H+L) + p\sin\zeta(H+L), & Z_h^+ &= \zeta\cosh\zeta(H+L) + p\sinh\zeta(H+L)\\
Z^0 &= \zeta\cos\zeta L + p\sin\zeta L, & Z_h^0 &= \zeta\cosh\zeta L + p\sinh\zeta L
\end{aligned}
$$

$$
\lambda^2 = \left(\frac{Z^-}{Z^+}\right)^2 = \left(\frac{Z_h^-}{Z_h^+}\right)^2.
$$

$$
\frac{Z^-}{Z^+} = \frac{Z_h^-}{Z_h^+} \to \zeta_1^k(p,H,L), \qquad \frac{Z^-}{Z^+} = -\frac{Z_h^-}{Z_h^+} \to \zeta_2^k(p,H,L),
$$

Nonoverlapping case :

$$
\begin{aligned}
F_1(\zeta) &= \sinh H\zeta\cos H\zeta - \sin H\zeta\cosh H\zeta\\
F_2(\zeta;p) &= \zeta^2\cos H\zeta\cosh H\zeta - p^2\sin H\zeta\sinh H\zeta.
\end{aligned}
$$

2×2 subdomains : dispersion relation

$$\frac{Z^-}{Z^+} = (-1)^i \frac{Z_h^-}{Z_h^+} \iff F_j(\zeta) = 0.$$

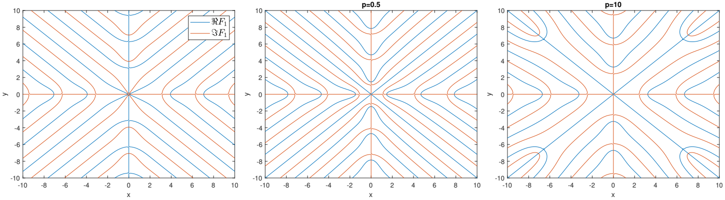
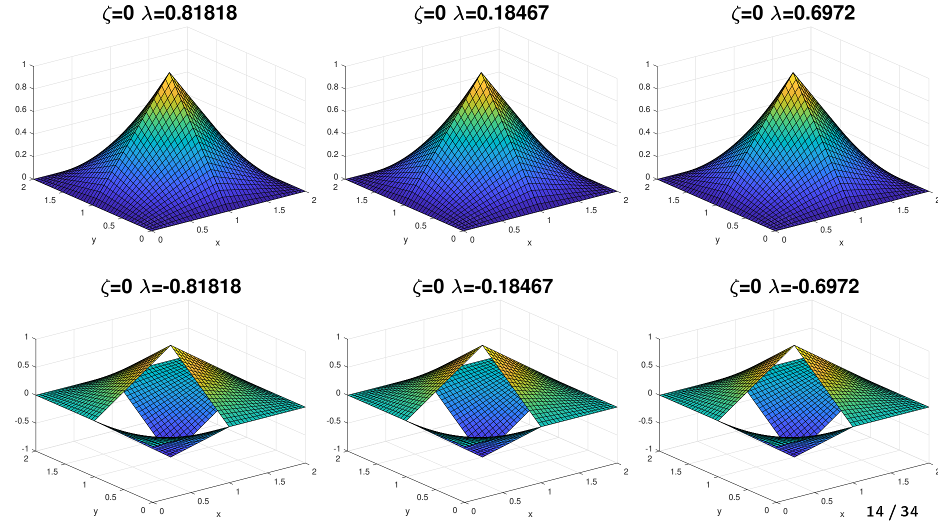


FIG. 3.1. $\Re F_j(x+iy) = 0$ *in red and* $\Im F_j(x+iy) = 0$ *in blue. Left* F_1, *middle* F_2 *with* $pH = 0.5$, *right* F_2 *with* $pH = 10$.

THEOREM 3.1. *F_1 and F_2 have a countable number of zeros. The zeros of F_1 are either real or pure imaginary. The zeros of F_2 are either real or pure imaginary, or diagonal, that is $\Re = \Im$, in the case $pH > 1$.*

2 × 2 subdomains : Affine modes

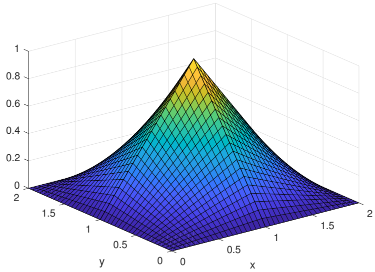
Classical Schwarz | Overlapping Robin ** | Nonoverlapping Robin



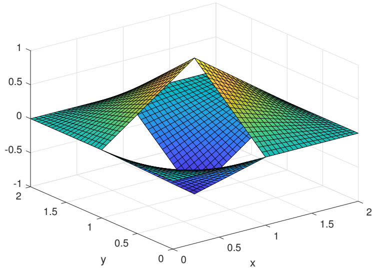
Fundamental modes

Theorem, 2020

When L is small (PS) and/or p is large (Robin), the affine modes for $N \times N$ subdomains are *asymptotically* special linear combinations of



Θ^C



Θ^D

Continuous/discontinous

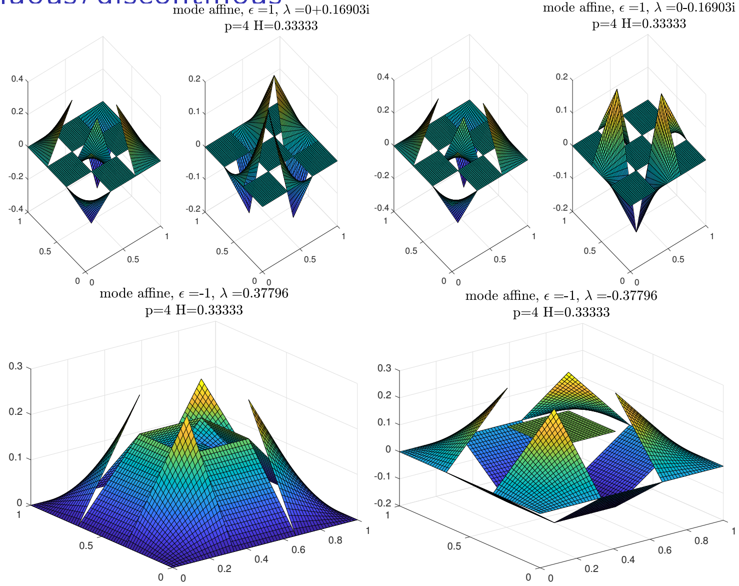
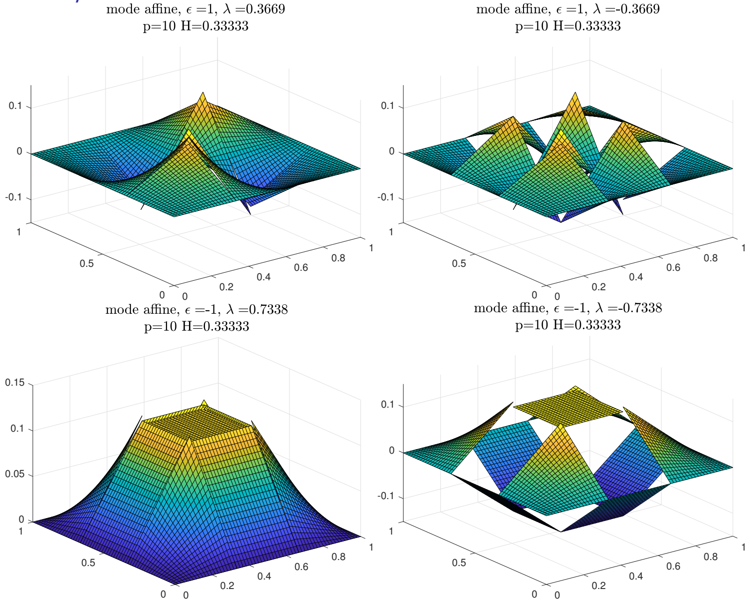
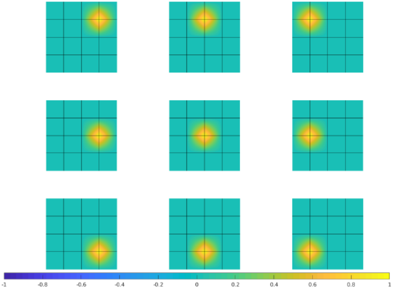


FIGURE 8.3 – 2 Modes affines réels pour 9 sous-domaines, $p = 4$, $H = 1/3$.

Continuous/discontinous

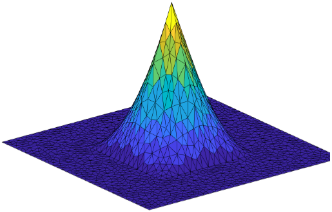


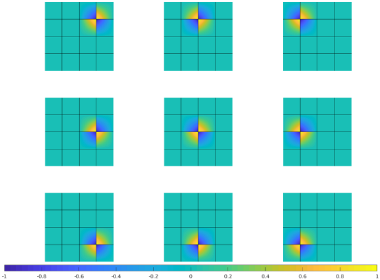
Coarse space assembly from the new basic elements



Q1FEM, continuous functions 1 to 9

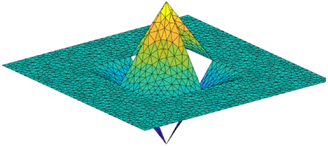
Q1FEM, continuous function (number 5)





Q1FEM, discontinuous functions 1 to 9

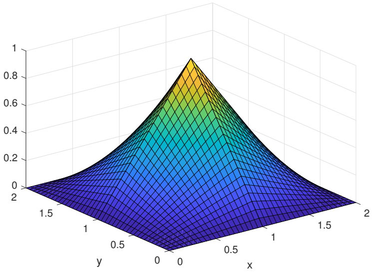
Q1FEM, discontinuous function (number 5)



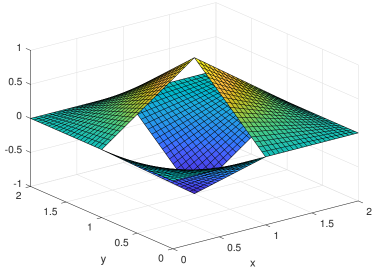
The two-level parallel Schwarz method

Principle of the new two-level parallel Schwarz method

1. smoothing by Schwarz with $\nu \geq 1$ smoothing steps(multigrid inspiration),
2. coarse correction, coarse space assembled from Θ^C and Θ^D,
3. use DCS-DMNV for the projection (Gander, Halpern, Santugini, On optimal coarse spaces for domain decomposition and their approximation, DD XXIV, Springer, 2017)
4. for Robin-Schwarz use optimized parameter.

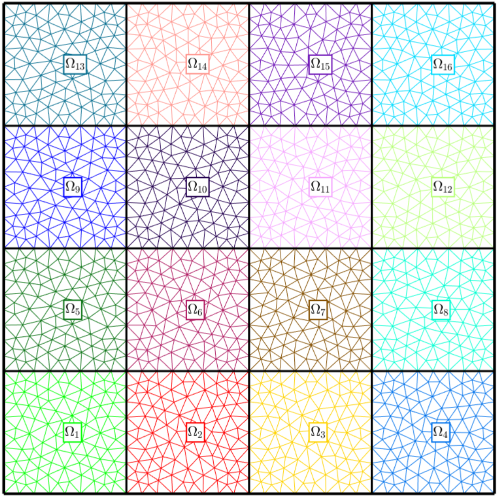


Θ^C



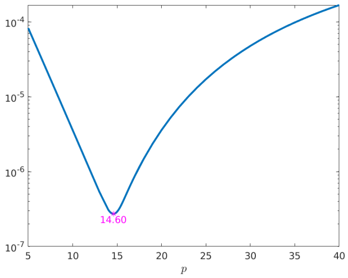
Θ^D

Two- level Experiment : 4 × 4 subdomains, FEM, no overlap

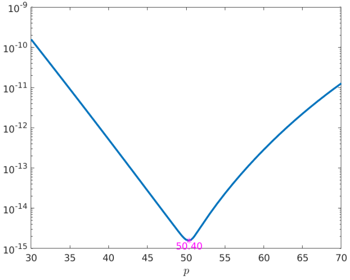


Optimized Robin-Schwarz, best parameter

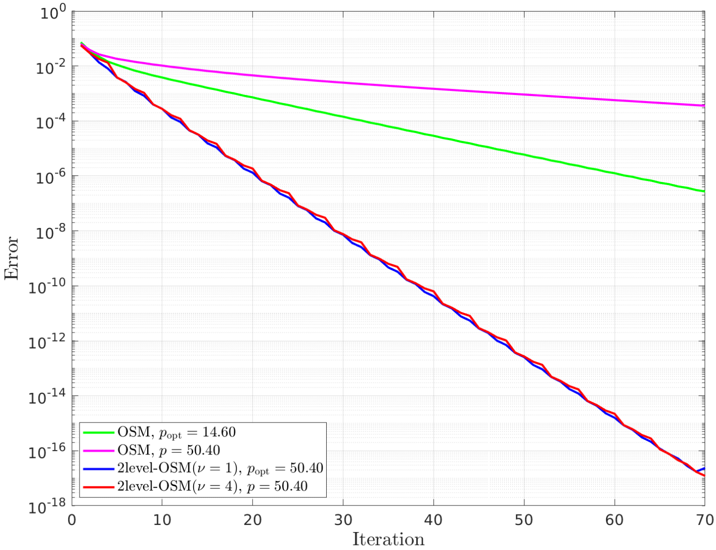
One-level Optimization



two-level Optimization



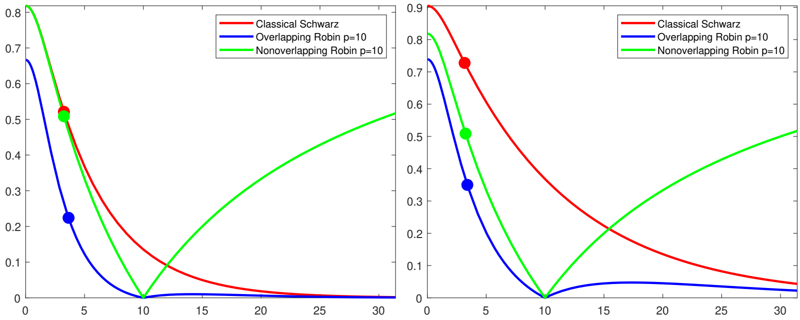
Optimized Robin-Schwarz, convergence



Optimization of the Robin coefficient

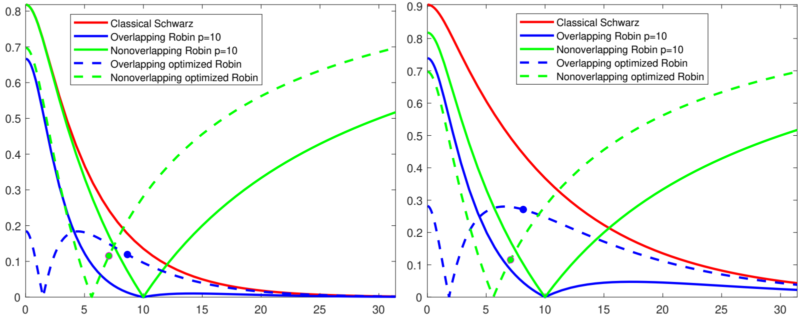
Convergence factor : 2 × 2 subdomains

Convergence factor : $|\lambda| = \left|\frac{Z_h^-}{Z_h^+}\right|$ as a function of ζ.



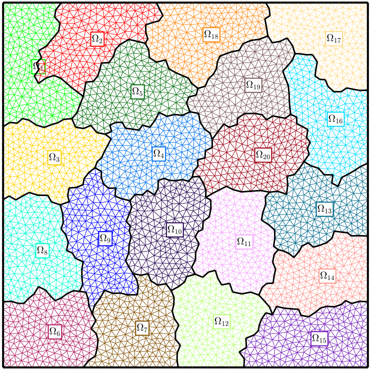
$L = 0.1$ $L = 0.05$

2 × 2 subdomains : optimized Robin

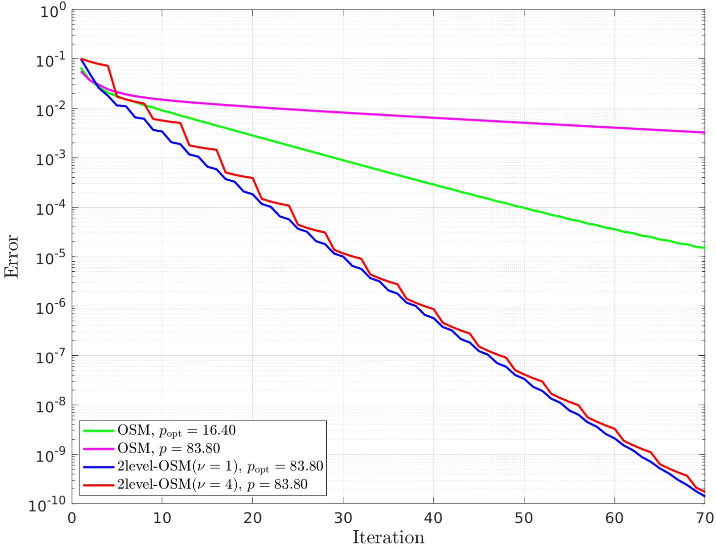


$L = 0.1$ $\qquad$ $L = 0.05$

General decomposition (METIS)



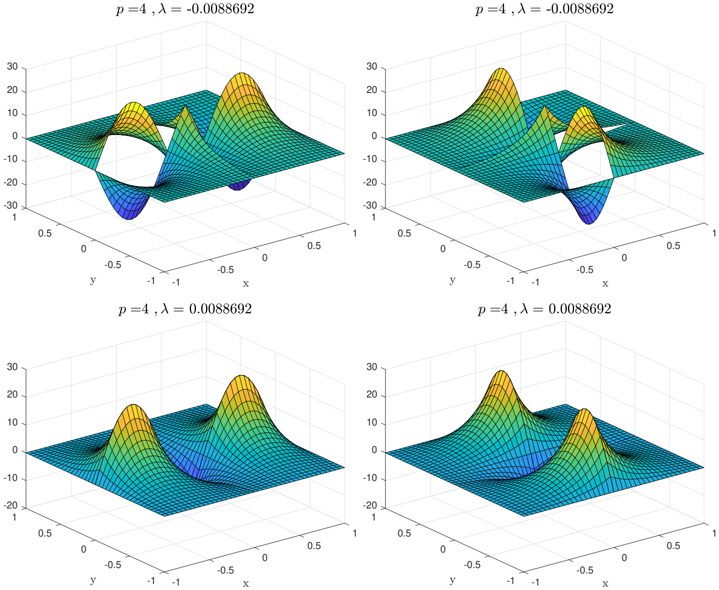
General decomposition (METIS)



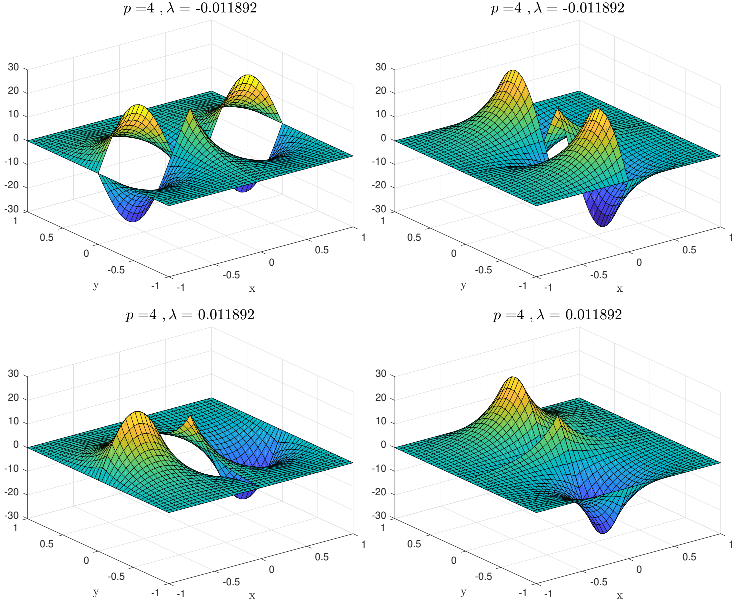
Two-level enrichment

Other modes for 2×2 subdomains

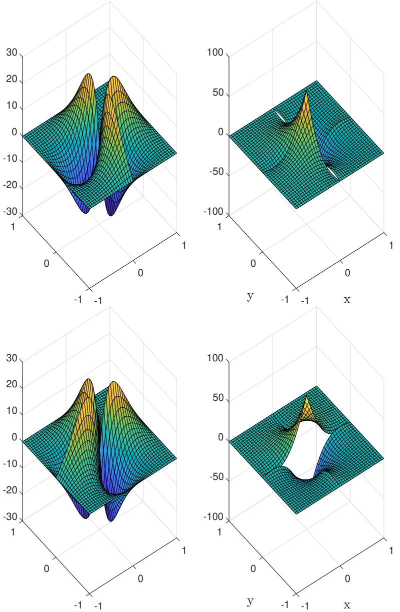
Modes of type I



Modes of type II

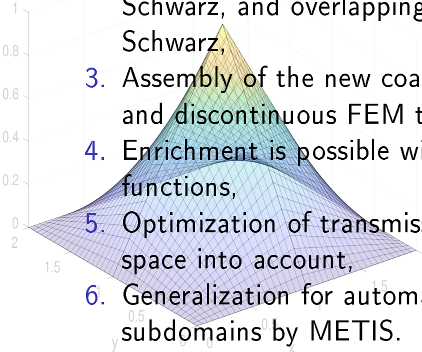


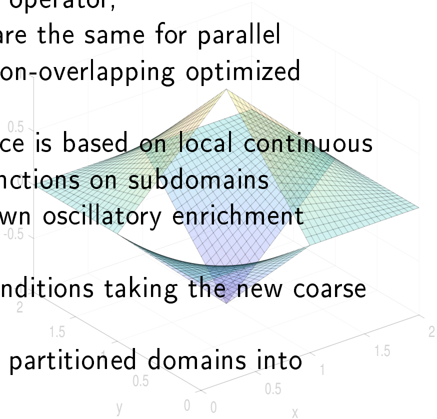
Diagonal mode



Summary

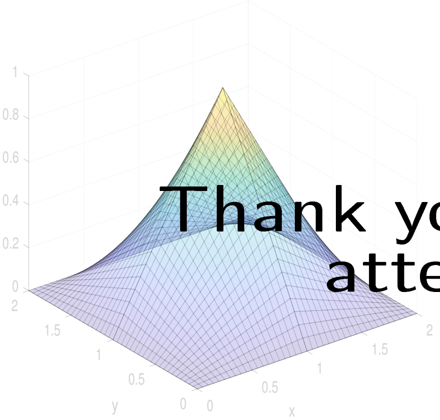
1. New coarse space components are based on a spectral analysis of the parallel Schwarz iteration operator,
2. New coarse space components are the same for parallel Schwarz, and overlapping and non-overlapping optimized Schwarz,
3. Assembly of the new coarse space is based on local continuous and discontinuous FEM type functions on subdomains
4. Enrichment is possible with known oscillatory enrichment functions,
5. Optimization of transmission conditions taking the new coarse space into account,
6. Generalization for automatically partitioned domains into subdomains by METIS.





Ongoing : Full analysis of the 2×2 operator.

Future work : assembly based new coarse space for high contrast problems, $\cdots$



Thank you for your attention