Optimized Schwarz algorithms on non overlapping grids for anisotropic elliptic operators in the framework of DDFV schemes

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work in collaboration with
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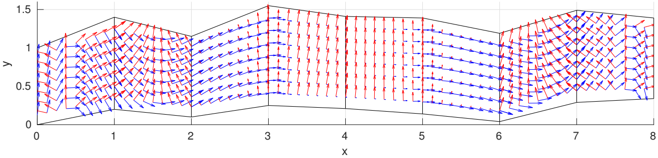
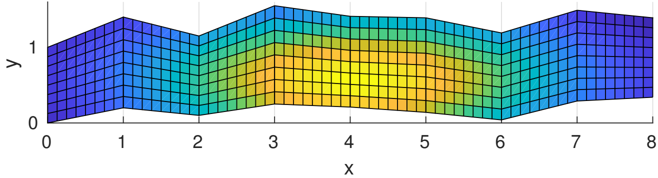
Some numerical challenges

Strong anisotropy on complex domain

$$-\text{div}(\mathbb{A}(x)\nabla u) = f \text{ with } \mathbb{A}(x) = a_n(x)\mathbf{n}\mathbf{n}^T + a_t(x)\mathbf{t}\mathbf{t}^T$$



In red $a_n\mathbf{n}$, in blue $a_t\mathbf{t}$



Mesh and source term

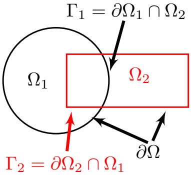
The domain that can naturally be split into subdomains

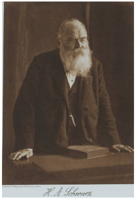
$\rightsquigarrow$ How behave the Schwarz methods in such a case ?

On Schwarz algorithms

Existence and uniqueness of harmonic functions on general domain

$$-\Delta u = 0 \text{ on } \Omega = \Omega_1 \cup \Omega_2,\ u = g \text{ on } \partial\Omega$$





1843-1921

Through an interative process

⇝ On the disk

$$\begin{cases} -\Delta u_1^l = 0 \text{ on } \Omega_1 \\ u_1^l = g \text{ on } \partial\Omega \cap \partial\Omega_1 \\ u_1^l = u_2^{l-1} \text{ on } \Gamma_1 \end{cases}$$

⇝ On the rectangle

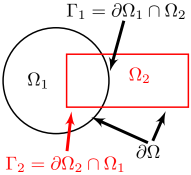
$$\begin{cases} -\Delta u_2^l = 0 \text{ on } \Omega_2 \\ u_2^l = g \text{ on } \partial\Omega \cap \partial\Omega_2 \\ u_2^l = u_1^l \text{ on } \Gamma_2 \end{cases}$$

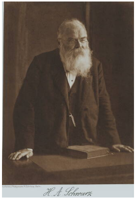
- called the classical **alternative** Schwarz method.
- Convergence proved by H. A. Schwarz in 1869 thanks to maximum principle argument.

On Schwarz algorithms

Existence and uniqueness of harmonic functions on general domain

$$-\Delta u = 0 \text{ on } \Omega = \Omega_1 \cup \Omega_2, \; u = g \text{ on } \partial\Omega$$





1843-1921

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⇝ On the rectangle

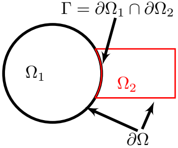
$$\begin{cases} -\Delta u_2^l = 0 \text{ on } \Omega_2 \\ u_2^l = g \text{ on } \partial\Omega \cap \partial\Omega_2 \\ u_2^l = u_1^{l-1} \text{ on } \Gamma_2 \end{cases}$$

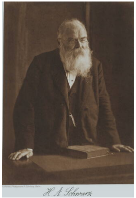
- called the classical **parallel** Schwarz method.
- convergence proved by P.L. Lions in 1988 thanks to Fourier expansion.

Schwarz algorithm for nonoverlapping subdomains

⇝ we will focus on nonoverlapping subdomains in this talk!

$$Lu = f \text{ on } \Omega = \cup\Omega_i,\ u = g \text{ on } \partial\Omega$$





1843-1921

The interative process (alternative or parallel)

⇝ On the disk

$$\begin{cases} Lu_1^l = f \text{ on } \Omega_1 \\ u_1^l = g \text{ on } \partial\Omega \cap \partial\Omega_1 \\ \partial_n u_1^l + \Lambda u_1^l = \partial_n u_2^{l-1} + \Lambda u_2^{l-1} \text{ on } \Gamma \end{cases}$$

⇝ On Ω_2

$$\begin{cases} Lu_2^l = f \text{ on } \Omega_2 \\ u_2^l = g \text{ on } \partial\Omega \cap \partial\Omega_2 \\ \partial_n u_2^l + \Lambda u_2^l = \partial_n u_1^{l-1} + \Lambda u_1^{l-1} \text{ on } \Gamma \end{cases}$$

Convergence

- Convergence for $L = -\Delta$ by P.L. Lions in 1990 by energy estimates

The classical transmission operators

- Robin transmission : $\Lambda u = pu$
- Ventcell transmission : $\Lambda u = pu + q\partial^2_{\tau\tau} u$

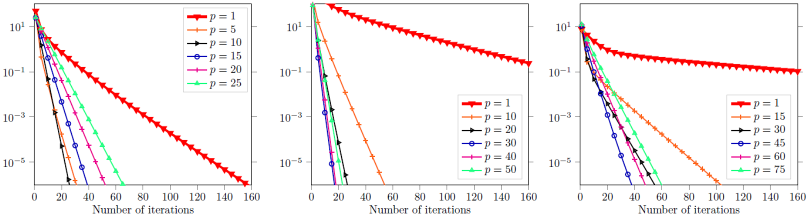
Numerical illustration-towards optimized parameters

Gander, Halpern, H. Krell MJPAA, 2021

Case of $-\text{div}(\mathbb{A}\nabla u) = f$ with Robin BC and $\mathbb{A}$ possibly anisotropic.

Deep influence of the choice of p - Square mesh $h = 2^{-3}$

$A_{xx} = A_{yy} = 1$ | $A_{xx} = 16,\ A_{yy} = 1$ | $A_{xx} = 1,\ A_{yy} = 16$



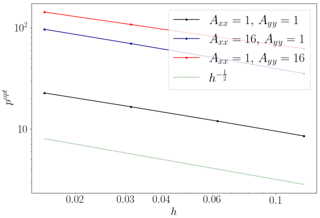
$p^{opt} \sim 10$ | $p^{opt} \sim 30$ | $p^{opt} \sim 45$

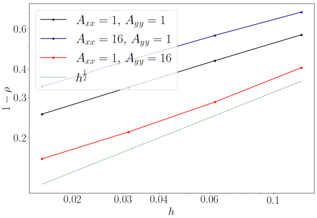
- ⇝ There exists an optimal choice p^{opt} for p!
- ⇝ The best parameter p^{opt} is strongly impacted by the anisotropy.

Numerical illustration-towards optimized parameters

Gander, Halpern, H. Krell MJPAA, 2021

Case of $-\mathrm{div}(\mathbb{A}\nabla u) = f$ with Robin BC and $\mathbb{A}$ possibly anisotropic.
Deep influence of the choice of p - influence of the mesh size





- ⇝ The best parameter is strongly impacted by the mesh size : $p^{opt} \sim Ch^{-\frac{1}{2}}$
- ⇝ The convergence factor is strongly impacted by the mesh size : $\rho^{opt} \sim 1 - Ch^{\frac{1}{2}}$

How can we estimate the optimal parameters for a general mesh ?

Outline

1. **Discrete Schwarz algorithm based on DDFV discretisation**
   - DDFV formalism
   - DDFV Schwarz algorithm

2. **Estimating the best parameters**
   - Optimization problem for the continuous problem
   - Optimization problem for the discrete problem

Outline

Classical finite volume strategy

The problem

$$-\mathrm{div}(\mathbb{A}\nabla u) = f, \quad \text{on } \Omega,\ u = 0 \text{ on } \partial\Omega$$

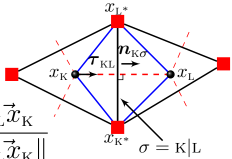
Main principle

- Let $\mathcal{T} = \cup \mathrm{K}$ a partition of Ω. Associate a point x_{K} to each $\mathrm{K} \in \mathcal{T}$.
- Integrate on any control volume K the equation :

$$-\int_{\mathrm{K}} \mathrm{div}(\mathbb{A}\nabla u)\, dx = -\sum_{\sigma\in\partial\mathrm{K}} \int_{\sigma} \mathbb{A}\nabla u \cdot \boldsymbol{n}_{\mathrm{K}\sigma} = \int_{\mathrm{K}} f(x)\, dx$$

- Approximate $\int_\sigma \mathbb{A}\nabla u \cdot \boldsymbol{n}_{\mathrm{K}\sigma}$ in a consistant and conservative way.

▶ Case $\mathbb{A} = Id$. Taylor expansion for $\sigma = \mathrm{K}|\mathrm{L}$



$$m_\sigma \frac{u(x_{\mathrm{L}}) - u(x_{\mathrm{K}})}{d_{\mathrm{KL}}} \sim \int_\sigma \nabla u \cdot \boldsymbol{\tau}_{\mathrm{KL}} \text{ where } \boldsymbol{\tau}_{\mathrm{KL}} = \frac{\vec{x_{\mathrm{L}} x_{\mathrm{K}}}}{\|\vec{x_{\mathrm{L}} x_{\mathrm{K}}}\|}$$

⇝ consistency requires orthogonality $\boldsymbol{\tau}_{\mathrm{KL}} = \boldsymbol{n}_{\mathrm{K}\sigma}$.

Classical finite volume strategy

The problem

$$-\mathrm{div}(\mathbb{A}\nabla u) = f, \quad \text{on } \Omega,\ u = 0 \text{ on } \partial\Omega$$

Main principle

- Let $\mathcal{T} = \cup \mathrm{K}$ a partition of Ω. Associate a point x_{K} to each $\mathrm{K} \in \mathcal{T}$.
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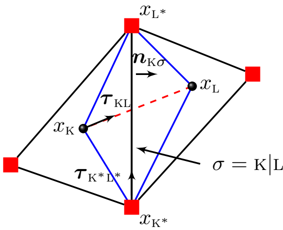
$$-\int_{\mathrm{K}} \mathrm{div}(\mathbb{A}\nabla u)\, dx = -\sum_{\sigma \in \partial \mathrm{K}} \int_{\sigma} \mathbb{A}\nabla u \cdot \boldsymbol{n}_{\mathrm{K}\sigma} = \int_{\mathrm{K}} f(x)\, dx$$

- Case $\mathbb{A} = Id$ + orthogonality condition
  $\boldsymbol{\tau}_{\mathrm{KL}} = \boldsymbol{n}_{\mathrm{K}\sigma}$
  $\rightsquigarrow$ FV4 scheme or TPFA scheme
- If not, we need to approximate the gradient in another direction (*e.g.* $\boldsymbol{\tau}_{\mathrm{K^*L^*}}$).
  $\rightsquigarrow$ *e.g.* DDFV scheme that requires unknowns at both centers x_{K} and vertices $x_{\mathrm{K^*}}$ of the control volumes.



DDFV strategy for an elliptic problem

The problem

$$-\mathrm{div}\,(\mathbb{A}(z)\nabla u(z)) = f(z) \quad \text{in } \Omega + BC$$

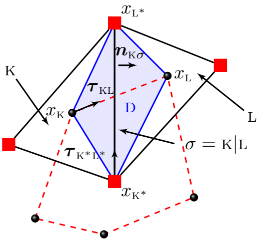
Main principle

- Consider a partition of Ω called **primal mesh**.
- Associate to each vertex a cell called **dual cell**.
- Integrate on both **primal cells** and **dual cells** the equation :

$$-\int_{\mathrm{C}} \mathrm{div}\,(\mathbb{A}(z)\nabla u(z))\,dz = -\sum_{\sigma\in\partial \mathrm{C}} \int_\sigma \mathbb{A}(z)\nabla u \cdot \boldsymbol{n}_{\mathrm{C}\sigma} = \int_{\mathrm{C}} f(z)\,dz$$

⇝ A natural way to approximate the divergence

- Approximate the normal fluxes $\int_\sigma \mathbb{A}(z)\nabla u \cdot \boldsymbol{n}_{\mathrm{C}\sigma}$
  - Define a **diamond cell** around each edges of the meshes $\mathrm{D} = (x_{\mathrm{K}}, x_{\mathrm{K}^*}, x_{\mathrm{L}^*}, x_{\mathrm{L}})$
  - Taylor expansion to approximate ∇u on each D using $(u(x_{\mathrm{K}}), u(x_{\mathrm{L}}), u(x_{\mathrm{K}^*}), u(x_{\mathrm{L}^*}))$

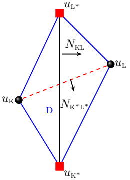


Discrete operators

Discrete gradient : $\nabla^{\mathcal{D}} : \mathbb{R}^{\mathcal{T}} \to (\mathbb{R}^2)^{\mathcal{D}}$ defined by

$$\nabla_{\mathrm{D}} u_{\mathcal{T}} = \frac{1}{2m_{\mathrm{D}}} \left((u_{\mathrm{L}} - u_{\mathrm{K}}) N_{\mathrm{KL}} + (u_{\mathrm{L}^*} - u_{\mathrm{K}^*}) N_{\mathrm{K}^*\mathrm{L}^*} \right).$$

- $N_{\mathrm{KL}} = (x_{\mathrm{L}^*} - x_{\mathrm{K}^*})^{\perp}$
- $N_{\mathrm{K}^*\mathrm{L}^*} = -(x_{\mathrm{L}} - x_{\mathrm{K}})^{\perp}$



$\nabla_{\mathrm{D}} u_{\mathcal{T}}$ is such that $\begin{cases} \nabla_{\mathrm{D}} u_{\mathcal{T}} \cdot (x_{\mathrm{L}} - x_{\mathrm{K}}) = u_{\mathrm{L}} - u_{\mathrm{K}}, \\ \nabla_{\mathrm{D}} u_{\mathcal{T}} \cdot (x_{\mathrm{K}^*} - x_{\mathrm{L}^*}) = u_{\mathrm{K}^*} - u_{\mathrm{L}^*}. \end{cases}$

The discrete divergence $\mathrm{div}^{\mathcal{T}} : (\mathbb{R}^2)^{\mathcal{D}} \to \mathbb{R}^{\mathcal{T}}$ defined by

$$\mathrm{div}_{\mathrm{K}}(\xi_{\mathcal{D}}) := \frac{1}{m_{\mathrm{K}}} \sum_{\mathrm{D} \in \mathcal{D}_{\mathrm{K}}} \xi_{\mathrm{D}} \cdot N_{\mathrm{KL}}, \quad \mathrm{div}_{\mathrm{K}^*}(\xi_{\mathcal{D}}) := \frac{1}{m_{\mathrm{K}^*}} \sum_{\mathrm{D} \in \mathcal{D}_{\mathrm{K}^*}} \xi_{\mathrm{D}} \cdot N_{\mathrm{K}^*\mathrm{L}^*}$$

Natural approx. of the divergence $\int_{\mathrm{K}} \mathrm{div}(\xi) = \sum_{\sigma = \mathrm{K}|\mathrm{L} \in \mathcal{D}_\sigma} \int_\sigma \xi(s) \cdot \boldsymbol{n}_{\mathrm{KL}}$!

The discrete duality property

Version for homogeneous Dirichlet BC

Lemma

For $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ *such that* $u_{\partial\mathfrak{M}} = 0$ *and* $u_{\partial\mathfrak{M}^*} = 0$ *and for* $\xi_{\mathcal{D}} \in (\mathbb{R}^2)^{\mathcal{D}}$,

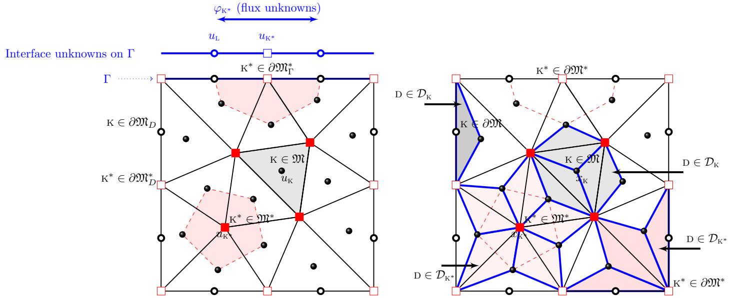
$$\left(\mathrm{div}^{\mathcal{T}} \xi_{\mathcal{D}}, u_{\mathcal{T}}\right) := \frac{1}{2} \sum_{\mathrm{K} \in \mathfrak{M}} m_{\mathrm{K}} \mathrm{div}_{\mathrm{K}}(\xi_{\mathcal{D}}) u_{\mathrm{K}} + \frac{1}{2} \sum_{\mathrm{K}^* \in \mathfrak{M}^*} m_{\mathrm{K}^*} \mathrm{div}_{\mathrm{K}^*}(\xi_{\mathcal{D}}) u_{\mathrm{K}^*}$$

$$= - \sum_{\mathrm{D} \in \mathcal{D}} m_{\mathrm{D}} \xi_{\mathrm{D}} \cdot \nabla_{\mathrm{D}} u_{\mathcal{T}} := - \int_{\Omega} \xi_{\mathcal{D}} \cdot \nabla^{\mathcal{D}} u_{\mathcal{T}}$$

A crucial tool to study the schemes!

DDFV meshes and unknowns

Primal, dual meshes (Left). Diamond meshes (Right)



Strategy

- Integrate the equation on both primal and dual cells $\rightsquigarrow$ a natural approximation of the divergence
- Define an approximate gradient on the diamond cells $\mathrm{D} \in \mathcal{D}$

Unknowns $(u_{\mathcal{T}}, \varphi_{\mathcal{T}}) \in \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\partial\mathfrak{M}^*_\Gamma}$

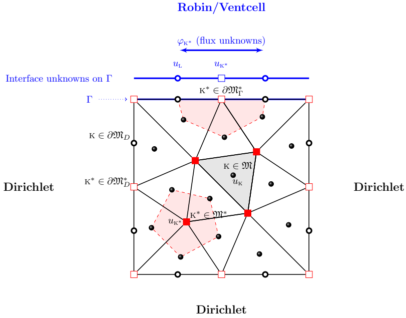
- One unknown per cell (called **primal**) $u_{\mathfrak{M}} = (u_{\mathrm{K}})_{\mathrm{K}\in\mathfrak{M}\cup\partial\mathfrak{M}}$
- One unknown per vertex (called **dual**) $u_{\mathfrak{M}^*} = (u_{\mathrm{K}^*})_{\mathrm{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}$
- Additionnal unknowns needed on $\partial\mathfrak{M}^*_\Gamma$

DDFV with mixed Robin or Ventcell/Dirichlet BC

$$\begin{cases} -\operatorname{div}(\mathbb{A}\nabla u) = f \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega_D \\ \dfrac{\partial u}{\partial n} + \Lambda u = g \text{ on } \Gamma \end{cases}$$

$$\Downarrow$$

$$\mathcal{L}^{\mathcal{T}}_{\Omega,\Gamma}(u_{\mathcal{T}}, \varphi_{\mathcal{T}}, f_{\mathcal{T}}, g_{\mathcal{T}}) = 0.$$



$-\operatorname{div}_C\left(\mathbb{A}^{\mathcal{D}}\nabla^{\mathcal{D}}u_{\mathcal{T}}\right) = f_C, \quad \forall\, C \in \mathfrak{M} \cup \mathfrak{M}^*,$ ⇝ Equations for primal cells and interior dual cells

$\sum_{D\in\mathcal{D}_{K^*}} \mathbb{A}^{\mathcal{D}}\nabla^{\mathcal{D}}u_{\mathcal{T}} \cdot N_{K^*\sigma} - \frac{1}{m_{K^*}}\varphi_{K^*} = f_{K^*}, \quad \forall\, K^* \in \partial\mathfrak{M}^*_\Gamma,$ ⇝ Equations for exterior dual cells

$u_C = 0, \quad \forall\, C \in \partial\mathfrak{M}_D \cup \partial\mathfrak{M}^*_D,$ ⇝ Dirichlet BC

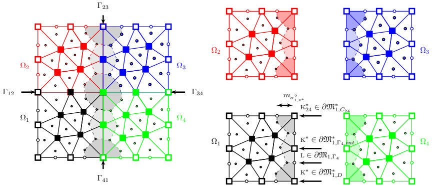
$\mathbb{A}^{\mathcal{D}}\nabla^{\mathcal{D}}u_{\mathcal{T}} \cdot N_{L\sigma} + \left(\Lambda^{\partial\mathfrak{M}_\Gamma} u_{\partial\mathfrak{M}_\Gamma}\right)_L = m_{\sigma_L} g_L, \quad \forall\, L \in \partial\mathfrak{M}_\Gamma,$ ⇝ Robin/Ventcell condition on primal boundary cells

$\varphi_{K^*} + \left(\Lambda^{\partial\mathfrak{M}^*_\Gamma} u_{\partial\mathfrak{M}^*_\Gamma}\right)_{K^*} = m_{\sigma_{K^*}} g_{K^*}, \quad \forall\, K^* \in \partial\mathfrak{M}^*_\Gamma,$ ⇝ Robin/Ventcell condition on dual boundary cells

DDFV Schwarz algorithm

Robin/Ventcell transmission condition

On each subdomain



- Choose $g^0_{\mathcal{T}_i}$.
- $\forall\ n \geq 0$, calculate

$$\mathcal{L}^{\mathcal{T}_j}_{\Omega_j,\Gamma}(u^{l+1}_{\mathcal{T}_j}, \varphi^{l+1}_{\mathcal{T}_j}, f_{\mathcal{T}_j}, g^l_{\mathcal{T}_j}) = 0.$$

- Then evaluate $g^{l+1}_{\mathcal{T}_i}$ by

$$g^{l+1}_{j,\mathrm{L}} = -\frac{1}{m_{\sigma_\mathrm{L}}} \mathbb{A}^{\mathcal{D}} \nabla^{\mathcal{D}} u^{l+1}_{\mathcal{T}_i} \cdot N_{\mathrm{KL}} + \Lambda^{\partial\mathfrak{M}_{i,j}}(u^{l+1}_i)_{\mathrm{L}},\ \forall\ \mathrm{L} \in \partial\mathfrak{M}_{j,\Gamma_i}$$

$$g^{l+1}_{j,\mathrm{K}^*} = -\frac{1}{m_{\sigma_{\mathrm{K}^*}}} \varphi^{l+1}_{i,\mathrm{K}^*} + \Lambda^{\partial\mathfrak{M}^*_{i,j}}(u^{l+1}_i)_{\mathrm{K}^*},\ \forall\ \mathrm{K}^* \in \partial\mathfrak{M}^*_{j,\Gamma_i}$$

$\rightsquigarrow$ Convergence of the algorithm proven by energy estimates

Outline

Best parameters from the continuous problem on $\mathbb{R}^2$

Assume that $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \mathbb{R}^- \times \mathbb{R}$ and $\Omega_2 = \mathbb{R}^+ \times \mathbb{R}$.
Consider for $\Lambda u = pu$ or $\Lambda u = pu - q\partial^2_{yy}u$ the problem

$\rightsquigarrow$ On Ω_1

$$\begin{cases} -\mathrm{div}(\mathbb{A}\nabla u_1^l) + \eta u_1^l = 0 \text{ on } \Omega_1 \\ u_1^l = 0 \text{ on } \partial\Omega \cap \partial\Omega_1 \\ \partial_n u_1^l + \Lambda u_1^l = \partial_n u_2^{l-1} + \Lambda u_2^{l-1} \text{ on } \Gamma \end{cases}$$

$\rightsquigarrow$ On Ω_2

$$\begin{cases} -\mathrm{div}(\mathbb{A}\nabla u_2^l) + \eta u_2^l = 0 \text{ on } \Omega_2 \\ u_2^l = 0 \text{ on } \partial\Omega \cap \partial\Omega_2 \\ \partial_n u_2^l + \Lambda u_2^l = \partial_n u_1^{l-1} + \Lambda u_1^{l-1} \text{ on } \Gamma \end{cases}$$

Find p^{opt} or (p^{opt}, q^{opt}) that leads to the faster $u_i^l \underset{l\to\infty}{\rightarrow} 0$.

- Case Robin — Gander SIAM JNA, 2006
- Case Ventcell — Gander, Halpern, H., Krell, MJPAA 2021

Best parameters from the continuous problem on $\mathbb{R}^2$

Assume that $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \mathbb{R}^- \times \mathbb{R}$ and $\Omega_2 = \mathbb{R}^+ \times \mathbb{R}$.
Consider for $\Lambda u = pu$ or $\Lambda u = pu - q\partial^2_{yy} u$ the problem

$\rightsquigarrow$ On Ω_1

$$\begin{cases} -\mathrm{div}(\mathbb{A}\nabla u_1^l) + \eta u_1^l = 0 \text{ on } \Omega_1 \\ u_1^l = 0 \text{ on } \partial\Omega \cap \partial\Omega_1 \\ \partial_n u_1^l + \Lambda u_1^l = \partial_n u_2^{l-1} + \Lambda u_2^{l-1} \text{ on } \Gamma \end{cases}$$

$\rightsquigarrow$ On Ω_2

$$\begin{cases} -\mathrm{div}(\mathbb{A}\nabla u_2^l) + \eta u_2^l = 0 \text{ on } \Omega_2 \\ u_2^l = 0 \text{ on } \partial\Omega \cap \partial\Omega_2 \\ \partial_n u_2^l + \Lambda u_2^l = \partial_n u_1^{l-1} + \Lambda u_1^{l-1} \text{ on } \Gamma \end{cases}$$

Find p^{opt} or (p^{opt}, q^{opt}) that leads to the faster $u_i^l \underset{l\to\infty}{\to} 0$.

Method : Fourier transform in the $y-$direction leads for all k to

$$-A_{xx}\frac{\partial^2 \hat{u}_j^l}{\partial x^2} - 2ikA_{xy}\frac{\partial \hat{u}_j^l}{\partial x} + (\eta + k^2 A_{yy})\hat{u}_j^l = 0 \quad + \quad BC$$

and looking for solutions on the form

$$\hat{u}_1^l(x,k) = C_1^l(k)e^{r_+(k)x}, \quad \hat{u}_2^l(x,k) = C_2^l(k)e^{r_-(k)x},$$

Results :

$$\begin{cases} C_j^l(k) = (\rho(P(k),k))^2\, C_j^{l-2}(k),\ \rho(P,k) := \frac{P(k;p,q) - f(k^2)}{P(k;p,q) + f(k^2)} \\ P(k;p,q) = p + qA_{yy}k^2,\ f(k^2) = \sqrt{\eta A_{xx} + k^2 \det \mathbb{A}} \end{cases}$$

Best parameters from the continuous problem on $\mathbb{R}^2$

Assume that $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \mathbb{R}^- \times \mathbb{R}$ and $\Omega_2 = \mathbb{R}^+ \times \mathbb{R}$.
Consider for $\Lambda u = pu$ or $\Lambda u = pu - q\partial^2_{yy} u$ the problem

$\leadsto$ On Ω_1

$$\begin{cases} -\mathrm{div}(\mathbb{A}\nabla u_1^l) + \eta u_1^l = 0 \text{ on } \Omega_1 \\ u_1^l = 0 \text{ on } \partial\Omega \cap \partial\Omega_1 \\ \partial_n u_1^l + \Lambda u_1^l = \partial_n u_2^{l-1} + \Lambda u_2^{l-1} \text{ on } \Gamma \end{cases}$$

$\leadsto$ On Ω_2

$$\begin{cases} -\mathrm{div}(\mathbb{A}\nabla u_2^l) + \eta u_2^l = 0 \text{ on } \Omega_2 \\ u_2^l = 0 \text{ on } \partial\Omega \cap \partial\Omega_2 \\ \partial_n u_2^l + \Lambda u_2^l = \partial_n u_1^{l-1} + \Lambda u_1^{l-1} \text{ on } \Gamma \end{cases}$$

Find p^{opt} or (p^{opt}, q^{opt}) that leads to the faster $u_i^l \underset{l\to\infty}{\to} 0$.

Results : The best parameter problem is reduced to an optimization problem on the form

$$\begin{cases} \text{Robin :} & \min_{(p)} \max_{\mu\in[\mu_{\min},\mu_{\max}]} \left|\frac{p-f(\mu)}{p+f(\mu)}\right| \\ \text{Ventcell :} & \min_{(p,q)} \max_{\mu\in[\mu_{\min},\mu_{\max}]} \left|\frac{p+qA_{yy}\mu-f(\mu)}{p+qA_{yy}\mu+f(\mu)}\right| \end{cases}$$

with $f(\mu) = \sqrt{\eta A_{xx} + \mu \det \mathbb{A}}$.

▶ From now on, we concentrate on the Robin case

Best parameters from the continuous problem on $\mathbb{R}^2$

Theorem

The optimization problem

$$p^{opt} = \text{Argmin}\left(\sup_{\mu=[\mu_{\min},\mu_{\max}]}\left|\frac{p-f(\mu)}{p+f(\mu)}\right|\right),$$

admits a unique solution given by $p^{opt} = \sqrt{f(\mu_{\max})f(\mu_{\min})}$, *and the associated convergence factor is*

$$\rho^{opt} = \left|\frac{\sqrt{f(\mu_{\max})}-\sqrt{f(\mu_{\min})}}{\sqrt{f(\mu_{\max})}+\sqrt{f(\mu_{\min})}}\right|.$$

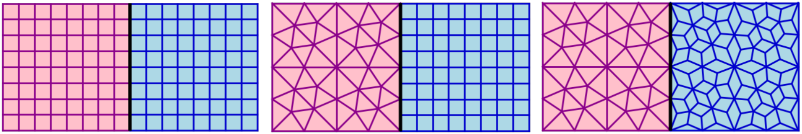
Here, $f(\mu) = \sqrt{\eta A_{xx} + \mu \det \mathbb{A}}$, $\mu = k^2$, $k_{\min} = 1$, $k_{\max}$ linked to the discretisation and the size of the domain.

$$\Rightarrow \begin{cases} p^*_\infty \sim \left(\eta A_{xx} + \left(\frac{\pi}{b}\right)^2 \det A\right)^{\frac{1}{4}} \left(\pi\sqrt{\det A}\right)^{\frac{1}{2}} h_y^{-\frac{1}{2}} \\ \rho^*_\infty \sim 1 - 2\left(\eta A_{xx} + \left(\frac{\pi}{b}\right)^2 \det A\right)^{\frac{1}{4}} \left(\pi\sqrt{\det A}\right)^{-\frac{1}{2}} h_y^{\frac{1}{2}} \end{cases}$$

Best parameters from the continuous problem on $\mathbb{R}^2$

How is the convergence for general meshes for these parameters ?

Example of meshes



Square-Square (ss) Square-Triangle (st) Triangle-Quadrangle (tq)

Numerical results (mesh size $h_y = \frac{1}{16}$, $k_{\max} = \frac{1}{h_y} - 1$)

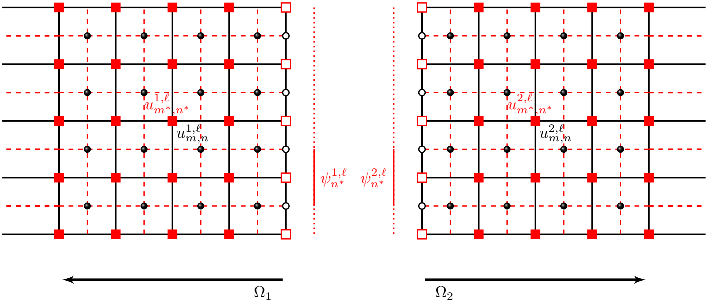
Best parameter type		Continuous study				Numerical study					
Problem			ss	ts	tq	ss		ts		tq	
A_{xx}	A_{yy}	$p^*_{\infty,\text{cvc}}$	$\breve{\rho}$	$\breve{\rho}$	$\breve{\rho}$	$\breve{p}^*$	$\breve{\rho}^*$	$\breve{p}^*$	$\breve{\rho}^*$	$\breve{p}^*$	$\breve{\rho}^*$
1	1	12.87	0.592	0.592	0.593	11.89	0.567	10.87	0.566	11.63	0.559
16	1	51.50	0.452	0.521	0.602	49.84	0.439	46.29	0.475	44.79	0.556
16	$\frac{1}{16}$	16.01	**0.351**	0.343	0.586	23.50	**0.174**	19.88	0.254	11.07	0.487
1	16	50.35	0.821	0.744	0.687	75.14	0.732	57.22	0.712	57.61	0.647
$\frac{1}{16}$	16	12.59	0.949	0.919	0.891	26.84	0.884	22.46	0.841	21.52	0.842

- ⇝ Good performance for the Laplace operator
- ⇝ Bad performance for anisotropic diffusion

Best parameters for infinite Cartesian grids

In the case of cartesian grids, **for a diagonal operator**, primal unknowns $(u^{j,l}_{m,n})$ and dual ones $(u^{j,l}_{m^*,n^*}, \psi^{j,l}_{n^*})$ are solution to two independant systems!

- j for the domain
- l for the iteration
- m or m^* for the horizontal position (in $\mathbb{N}$)
- n or n^* for the vertical position (in $\{1, \cdots, \frac{b}{hy}\}$)



The two systems differ in the neighbourhood of $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ leading to **different optimal parameters**!

Best parameters for infinite Cartesian grids

Both families $(u^{j,l}_{m,n})$ and dual ones $(u^{j,l}_{m^*,n^*})$ solves the recursion like

$$\frac{A_{xx}}{h_x^2}(v_{m+1,n}-2v_{m,n}+v_{m-1,n})+\frac{A_{yy}}{h_y^2}(v_{m,n+1}-2v_{m,n}+v_{m,n-1})-\eta v_{m,n}=0$$

Using Fourier expansion in the $y-$direction, we look for solution of the form

$$u^{j,\ell}_{m,n}=\sum_{k=1}^{k_{max}}\hat{u}^{j,\ell}_{m,k}\sin\left(k\pi n\frac{h_y}{b}\right),\ u^{j,\ell}_{m^*,n^*}=\sum_{k=1}^{k^*_{\max}}\hat{u}^{j,\ell}_{m^*,k}\sin\left(k\pi n^*\frac{h_y}{b}\right)$$

with

$$\hat{u}^{j,\ell}_{m,k}=C^{j,\ell}(k)\lambda(k)^m,\quad \hat{u}^{j,\ell}_{m^*,k}=D^{j,\ell}(k)\lambda(k)^{m^*}$$

Transmission conditions leads to

$$C^{j,l}=\rho_{cc,\infty}C^{j,l-2},\ D^{j,l}=\rho_{vc,\infty}D^{j,l-2}$$

where

$$\rho_{cc,\infty}(k;p)=\frac{p-f_{cc,\infty}(\nu(k))}{p+f_{cc,\infty}(\nu(k))},\rho_{vc,\infty}(k;p)=\frac{p-f_{vc,\infty}(\nu(k))}{p+f_{vc,\infty}(\nu(k))}$$

with $\begin{cases} f_{cc,\infty}(\nu(k))=2\frac{A_{xx}}{h_x}\tanh\left(\frac{\nu(k)}{2}\right)\\ f_{vc,\infty}(\nu(k))=\frac{A_{xx}}{h_x}\sinh(\nu(k))\end{cases}$ and $\begin{cases}\lambda(k):=1+\frac{\mu(k)}{2}-\sqrt{\mu(k)+\frac{\mu(k)^2}{4}}<1\\ \mu(k)=\frac{h_x^2}{A_{xx}}\left(\frac{4A_{yy}}{h_y^2}\sin^2(\frac{k\pi h_y}{2b})+\eta\right)\\ \nu(k)=-\ln(\lambda(k))\end{cases}$.

Best parameters for infinite Cartesian grids

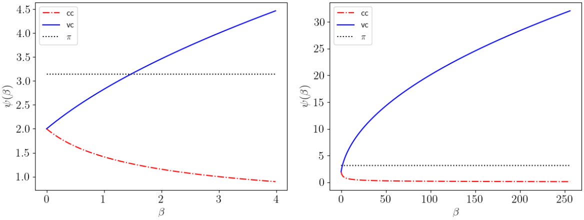
Optimization independently on primal and dual meshes

$$\begin{cases} p^{opt}_{cc,\infty} = \text{Argmin}_p \left(\sup_{k=1,\cdots,k_{\max}} \left| \frac{p - f_{cc,\infty}(\nu(k))}{p + f_{cc,\infty}(\nu(k))} \right| \right) \\ p^{opt}_{vc,\infty} = \text{Argmin}_p \left(\sup_{k=1,\cdots,k^*_{\max}} \left| \frac{p - f_{vc,\infty}(\nu(k))}{p + f_{vc,\infty}(\nu(k))} \right| \right) \end{cases}$$

with $k_{max} = \frac{b}{h_y}$ and $k^*_{\max} := \frac{b}{h_y} - 1$.

As in the continuous analysis

$$\begin{cases} p^{opt}_* \sim \psi_*^{\frac{1}{2}} \left(\eta A_{xx} + \left(\frac{\pi}{b}\right)^2 \det A \right)^{\frac{1}{4}} \left(\pi \sqrt{\det A} \right)^{\frac{1}{2}} h_y^{-\frac{1}{2}} \\ \rho^{opt}_* \sim 1 - 2\psi_*^{-\frac{1}{2}} \left(\eta A_{xx} + \left(\frac{\pi}{b}\right)^2 \det A \right)^{\frac{1}{4}} \left(\pi \sqrt{\det A} \right)^{-\frac{1}{2}} h_y^{\frac{1}{2}} \end{cases}$$



where $\beta := \frac{A_{yy}}{h_y^2} \frac{h_x^2}{A_{xx}}$

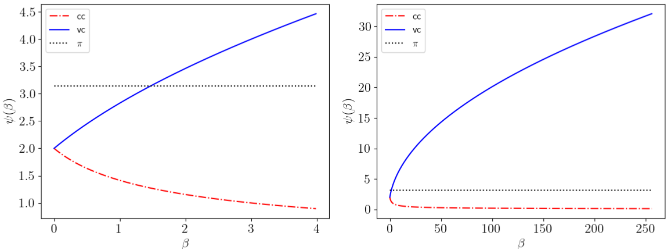
Best parameters for bounded domain

Similar results for bounded cartesian grids $[-a, a] \times [0, b]$.

Optimal parameters and convergence factors

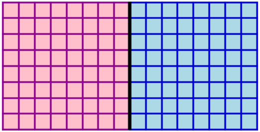
$$\begin{cases} p_*^{opt} \sim \psi_*^{\frac{1}{2}} \left(\eta A_{xx} + \left(\frac{\pi}{b}\right)^2 \det A\right)^{\frac{1}{4}} \left(\pi\sqrt{\det A}\right)^{\frac{1}{2}} c^{\frac{1}{2}} h_y^{-\frac{1}{2}} \\ \rho_*^{opt} \sim 1 - 2\psi_*^{-\frac{1}{2}} \left(\eta A_{xx} + \left(\frac{\pi}{b}\right)^2 \det A\right)^{\frac{1}{4}} \left(\pi\sqrt{\det A}\right)^{-\frac{1}{2}} c^{\frac{1}{2}} h_y^{\frac{1}{2}} \end{cases}$$

with $c = \coth\left(\frac{a}{\sqrt{A_{xx}}}\sqrt{\eta + \left(\frac{\pi}{b}\right)^2 A_{yy}}\right)$.

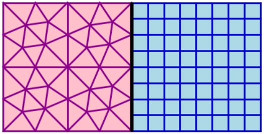


Best parameters for bounded Cartesian grids

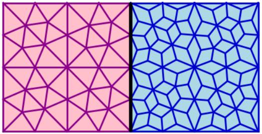
How is the convergence for general meshes for these parameters ?



Square-Square (ss)

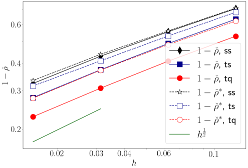


Square-Triangle (st)

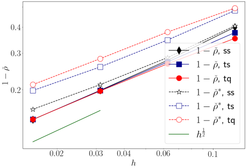


Triangle-Quadrangle (tq)

Discrete study - bounded domain



$A_{xx} = 16,\ A_{yy} = 1$



$A_{xx} = 1,\ A_{yy} = 16$

- For uniform meshes, the discrete study enable us to recover good convergence factors.
- These optimal parameters performs relatively well for general meshes.
- With adapted anisotropic meshes, we can use the continuous param.

Thank you for your attention!