Dynamics of a non-spherical microcapsule in shear flow

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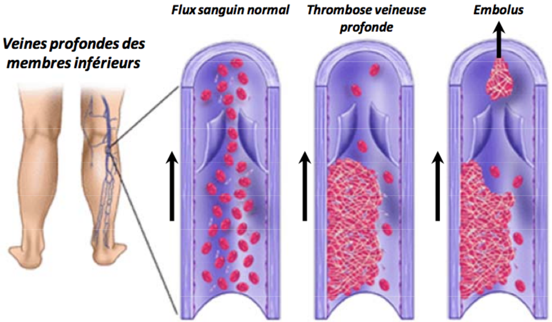
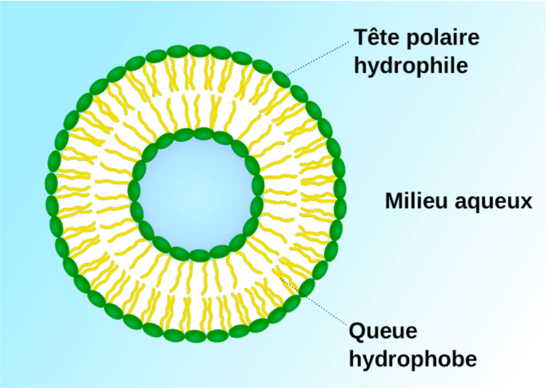
This is a joint interdisciplinary work with M. Guedda (LAMFA) and C. Misbah (LIPHY, Grenoble).

Motivation: physical properties of red blood cells

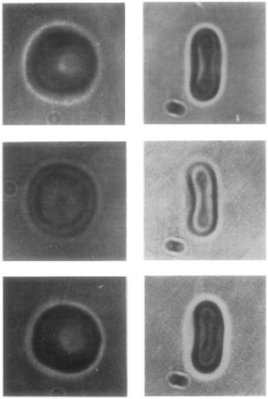


Red Blood Cell

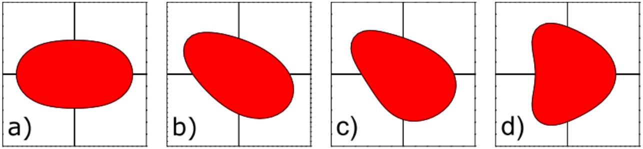
Vesicule



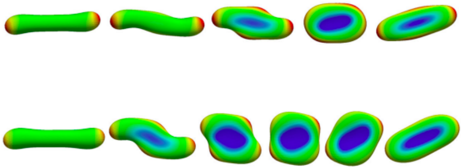
Under the microscope :



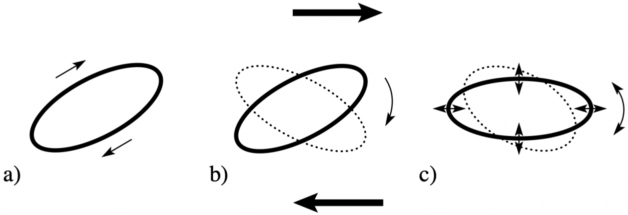
Particles have different shapes under shear flow.



Particles have different shapes under shear flow.



Three classical regimes :



Different regimes for quasi-spherical vesicles.

Contents

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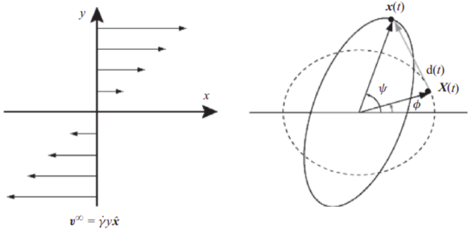
Mathematical modeling : small deformation approach



(P.M. Vlahovska et al., J. Fluid Mech 2011)

In a coordinate system centred at the capsule, r the radial position of the interface can be represented as

$$r = 1 + \epsilon f(\psi, \phi, t),$$

where ϵ is a small parameter, and f is the deviation of particle shape from a sphere of equivalent volume.

The function f is expanded into series of Y_{jm} scalar spherical harmonics

$$f = \sum_{j=2}^{\infty} \sum_{m=-j}^{j} f_{jm} Y_{jm},$$

$$f = f_{2-2} Y_{2-2} + f_{20} Y_{20} + f_{22} Y_{22}.$$

For $m = -2, 0, 2$,

$$\begin{cases} \frac{df_{2m}}{dt} = -i\frac{m}{2}\Lambda^{-1}\delta_{|m|2} + i\frac{m}{2}f_{2m} - \Lambda^{-1}\sqrt{\frac{\Delta}{30\pi}}[3\mathcal{B}^{-1}(6+\sigma_0)f_{2m} \\ \quad +2Ca^{-1}(f_{2m} - g_{2m})], \\ \frac{dg_{2m}}{dt} = i\frac{m}{2}g_{2m}, \\ 2|f_{22}|^2 + |f_{20}|^2 = 2. \end{cases}$$

$\Lambda = \frac{\sqrt{\Delta}(23(\mu_{in}/\mu_{ex})+32)}{8\sqrt{30\pi}}$, Δ is the excess area.

Setting

$$f_{2\pm 2} = R\, exp(\mp 2i\psi),$$

we get [1]

$$\frac{d\psi}{dt} = -\frac{1}{2} + \frac{\Lambda^{-1}}{2R}\cos(2\psi) + \epsilon_0 \frac{(S\Lambda)^{-1}}{2R}\sin(2\phi - 2\psi),$$

$$\frac{dR}{dt} = (S\Lambda)^{-1}\epsilon_0(1-R^2)\cos(2\phi - 2\psi) - (S\Lambda)^{-1} R\sqrt{(1-R^2)(1-\epsilon_0^2)} + \Lambda^{-1}(1-R^2)\sin(2\psi),$$

where $S^{-1} = \sqrt{\frac{2\Delta}{15\pi}} Ca^{-1}$ and $\frac{d\phi}{dt} = -1/2$.

[1] Vlahovska et al., JFM 2011

Mathematical modeling : analytical study

We introduce

$$f_{2\pm2} = \xi \mp i\kappa, \qquad g_{2\pm2} = a \mp i\, b,$$

$$\frac{dp}{dt} = \Omega.p - \Lambda^{-1}[j - (j.p)p] + (\Lambda S)^{-1}[A - (A.p)p)],$$

where

$$p = \begin{pmatrix} \xi \\ \kappa \\ \frac{f_{20}}{\sqrt{2}} \end{pmatrix}, \quad A = \begin{pmatrix} a \\ b \\ \frac{g_{20}}{\sqrt{2}} \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\Omega = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

General problem to solve

We consider the general problem

$$\frac{dp}{dt} = \Omega.p + j^* - (j^*.p)p,$$

with $j^* = \Lambda^{-1} j + S^{-1} A$, and

$$\Omega = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} a \\ b \\ \frac{g_{20}}{\sqrt{2}} \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

J.O.Kessler Hydrodynamic focusin of motile alga cells, Nature (1985).

General problem to solve

In Part 1, we will discuss a non-spherical particle,
in Part 2, we will present motion and trajectory of a spherical (rigid) particle.

Contents

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To solve

$$\frac{dp}{dt} = \Omega.p + j^* - (j^*.p)p,$$

with $S^{-1} << 1$, we use the following ansatz (approximation)

$$p = p_0 + S^{-1}\tilde{p},$$

and we obtain

$$\begin{cases} 0 = \Omega p_0 + \Lambda^{-1}[j - (j.p_0)p_0], \\ \frac{d\tilde{p}}{dt} = \tilde{\Omega}\tilde{p} + A - (A.p_0)p_0, \end{cases}$$

where

$$\tilde{\Omega} = \Omega - \Lambda^{-1}[jp_0^T + (j.p_0)I_d].$$

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Exact expression for $\tilde{p}$

For $\Lambda < 1$,

$$\tilde{p}_x = C_1 \exp(-\sqrt{\Lambda^{-2} - 1}\,t) + \Lambda\sqrt{1 - \Lambda^2} c \cos(t + \tilde{\theta}),$$

$$\tilde{p}_y = C_2 \exp(-2\sqrt{\Lambda^{-2} - 1}\,t) - \frac{C_1}{\sqrt{\Lambda^{-2}-1}} \exp(-\sqrt{\Lambda^{-2} - 1}\,t) - \Lambda^2 c \cos(t + \tilde{\theta}),$$

$$\tilde{p}_z = C_3 \exp(-\sqrt{\Lambda^{-2} - 1}\,t) + \frac{1}{\sqrt{\Lambda^{-2}-1}} \frac{g_{20}}{\sqrt{2}}.$$

Rocking motion

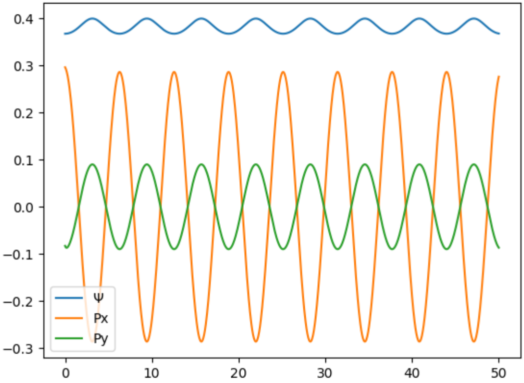
For $S^{-1} << 1$,

$$p \approx \begin{pmatrix} \Lambda \\ \sqrt{1-\Lambda^2} \\ \frac{S^{-1} g_{20}}{\sqrt{2}\sqrt{\Lambda^{-2}-1}} \end{pmatrix} + \Lambda S^{-1} c. \cos(t+\tilde{\theta}) \begin{pmatrix} \sqrt{1-\Lambda^2} \\ -\Lambda \\ 0 \end{pmatrix},$$

$$\psi \approx \frac{1}{2} \arccos \Lambda - \frac{\Lambda S^{-1}}{2} c. \cos(t+\tilde{\theta}),$$

(recall $f_{2\pm 2} = R \exp(\mp 2i\psi)$).

Rocking motion



Oscillating motion around VB state

For $\Lambda > 1$

$$p = \begin{pmatrix} \Lambda^{-1} \\ \frac{(S\Lambda)^{-1} g_{20}}{\sqrt{2}\sqrt{1-\Lambda^{-2}}} \\ \sqrt{1-\Lambda^{-2}} \end{pmatrix}$$

$$+\delta S^{-1} \cos(\sqrt{1-\Lambda^{-2}}t + \theta_0) \begin{pmatrix} 1 \\ 0 \\ \frac{\Lambda^{-1}}{\sqrt{1-\Lambda^{-2}}} \end{pmatrix}$$

$$+[\frac{S^{-1}\delta}{\sqrt{1-\Lambda^{-2}}} \sin(\sqrt{1-\Lambda^{-2}}t + \theta_0) - S^{-1} c. \cos(t + \tilde{\theta})]j.$$

Oscillating motion around VB state

For $S^{-1} << 1$,

$$\psi \approx \frac{1}{2}\frac{S^{-1}g_{20}}{\sqrt{1-\Lambda^{-2}}} + \frac{1}{2}\frac{\Lambda S^{-1}\delta}{\sqrt{1-\Lambda^{-2}}}\sin\left(\sqrt{1-\Lambda^{-2}}\,t+\theta_0\right) - \frac{1}{2}\Lambda S^{-1}c.\cos\left(t+\tilde{\theta}\right),$$

(recall $f_{2\pm 2} = R\ \exp(\mp 2i\psi)$).

$$\psi_{av} \approx \frac{1}{2}\frac{S^{-1}g_{20}}{\sqrt{1-\Lambda^{-2}}}$$

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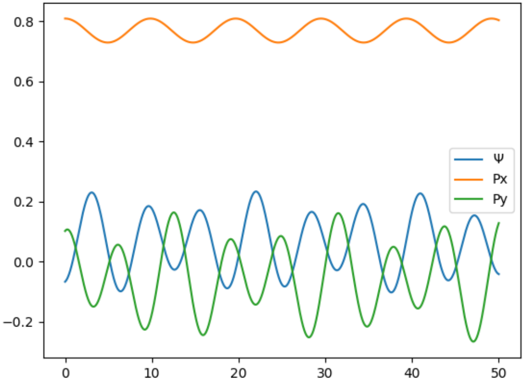
Oscillating motion around VB state



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Contents

Spherical (rigid) particle

We consider the general problem

$$\frac{dp}{dt} = \omega \times p + j^* - (j^*.p)p,$$

where j^* is the external force field, ω is the fluid vorticity vector (fictive).

The position equation is

$$\frac{dX}{dt} = v_s p + u_0.$$

where u_0 is the fluid velocity.

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Particle orientation (ω and j^* are perpendicular)

To solve

$$\frac{dp}{dt} = \Omega.p + j^* - (j^*.p)p,$$

we use the anstaz

$$p = p_0 + \alpha(t)q,$$

where p_0, α, and q satisfy

$$\begin{cases} 0 & = & \Omega p_0 + j^* - (j^*.p_0)p_0, \\ \dot{q} & = & \Omega q - (j^*.q)p_0, \\ \dot{\alpha} & = & -\alpha(j^*.p_0) - \alpha^2(j^*.q). \end{cases}$$

Trajectory of a spherical microswimmer

Therefore, any solution can be written as

$$p(t) = p_0 + \frac{\exp(\tilde{\Omega}_0 t) q(0)}{\alpha^{-1}(0) + \int_0^t (j^* . \exp(\tilde{\Omega}_0 s)) q(0) \, \mathrm{d}s},$$

where

$$\tilde{\Omega}_0 = \Omega - p_0 {j^*}^T,$$

and

$$p_0 = \xi \omega + \lambda \omega^*(\epsilon) \times j^*,$$

where $\xi \in \mathbb{R}$ and $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$, s.t. $\epsilon_1 + \epsilon_2 + \epsilon_3 = 1$, $\omega^*(\epsilon) = (\epsilon_1/\omega_1, \epsilon_2/\omega_2, \epsilon_3/\omega_3)$ is the vorticity vector.

Swimming direction (Run regime)

$u_0 = (y, 0, 0)$

For $\Lambda < 1$,

$$\begin{cases} p_x(t) = \Lambda^{-1} - \frac{\Lambda^{-2}-1}{\Lambda^{-1}} \frac{\cosh\left(\sqrt{\Lambda^{-2}-1}\,t\right)}{a+\cosh\left(\sqrt{\Lambda^{-2}-1}\,t\right)}, \\ p_y(t) = \frac{\sqrt{\Lambda^{-2}-1}}{\Lambda^{-1}} \frac{\sinh\left(\sqrt{\Lambda^{-2}-1}\,t\right)}{a+\cosh\left(\sqrt{\Lambda^{-2}-1}\,t\right)}, \\ p_z(t) = \frac{\sqrt{\Lambda^{-2}-1}}{\Lambda^{-1}} \frac{C}{a+\cosh\left(\sqrt{\Lambda^{-2}-1}\,t\right)}. \end{cases}$$

$$p \to p_{eq} = \Lambda i + \sqrt{1-\Lambda^2}\,j.$$

Run trajectory (at $p = p_{eq}$)

Therefore,

$$\begin{aligned} x(t) &= x(0) + (v_s \Lambda + y(0))t + \tfrac{v_s}{2}\sqrt{1-\Lambda^2}\,t^2, \\ y(t) &= y(0) + v_s\sqrt{1-\Lambda^2}\,t, \\ z(t) &= z(0). \end{aligned}$$

For the long-time

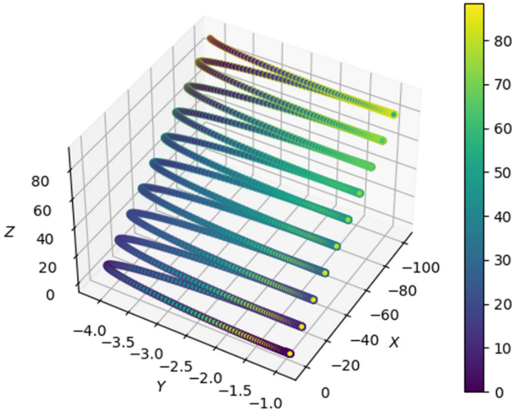
$$x \approx \frac{\Lambda^{-1}}{2v_s\sqrt{1-\Lambda^2}} y^2$$

Swimming direction (Tumbling regime)

For $\Lambda > 1$,

$$\begin{cases} p_x(t) = \Lambda^{-1} + \dfrac{1-\Lambda^{-2}}{\Lambda^{-1}} \dfrac{\sin\left(\sqrt{1-\Lambda^{-2}}t\right)}{a+\sin\left(\sqrt{1-\Lambda^{-2}}t\right)}, \\ p_y(t) = \dfrac{\sqrt{1-\Lambda^{-2}}}{\Lambda^{-1}} \dfrac{\cos(\sqrt{1-\Lambda^{-2}}t)}{a+\sin\left(\sqrt{1-\Lambda^{-2}}t\right)}, \\ p_z(t) = \dfrac{\sqrt{1-\Lambda^{-2}}}{\Lambda^{-2}} \dfrac{C}{a+\sin\left(\sqrt{1-\Lambda^{-2}}t\right)}. \end{cases}$$

Tumbling trajectory



Contents

Prospect

A rigid particle with a non-spherical shape.

The motion and transport of a red blood cell with Poiseuille flow (exact solution).

Thank You !