A new interpolation method for splines of any degree

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Presentation

- Discretize the interval $[0, T]$ in N sub-intervals of same length Δt

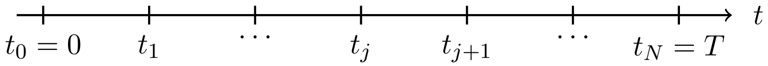
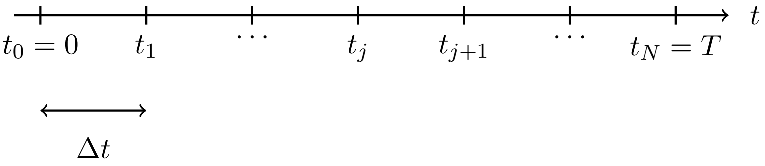


Figure: Interpolation nodes on the considered mesh.

- Discretize the interval $[0, T]$ in N sub-intervals of same length Δt



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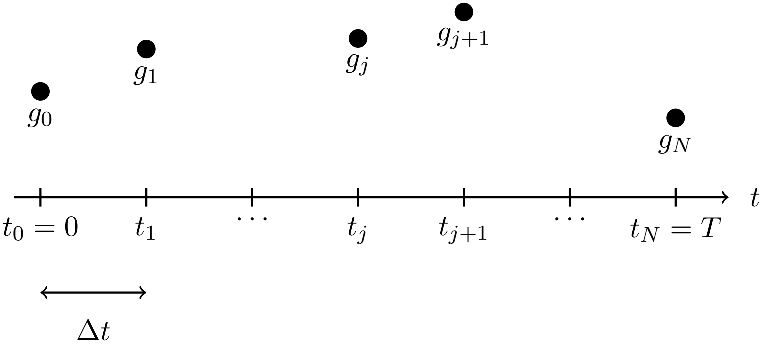
- Discretize the interval $[0, T]$ in N sub-intervals of same length Δt
- $N + 1$ interpolation nodes $t_0, t_1, \ldots, t_N$ with values $g_0, g_1, \ldots, g_N$



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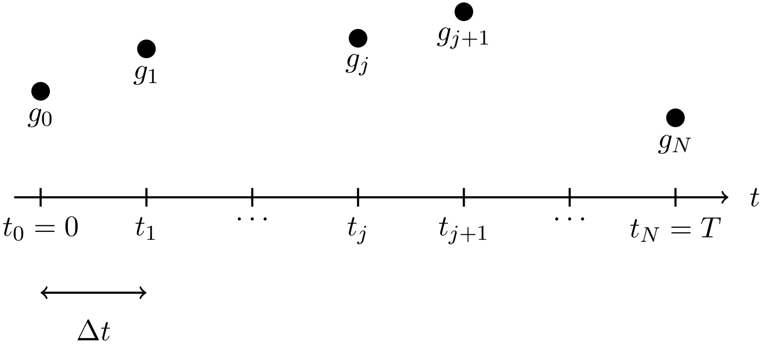
- Goal: Reconstruct the unknown function g using splines



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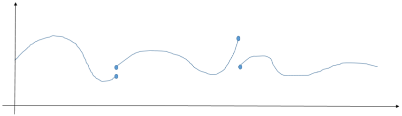
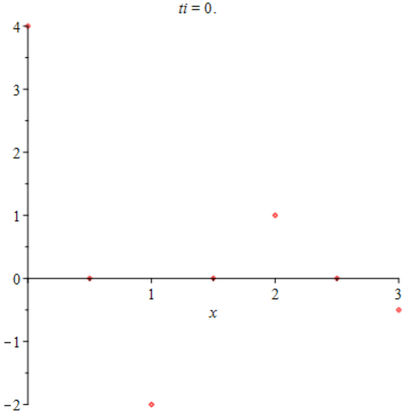


Figure: Example of a function that is not a spline.

Figure: Example of a cubic spline with natural boundary conditions [De Boor, 1978]



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- [Pepin et al., 2022] Algorithms to compute the boundary conditions b_n. New

Linear system to solve:

$$\begin{bmatrix} J_{1,k} & J_{2,k} & \dots & J_{\theta,k} \\ J_{0,k} & J_{1,k} & \dots & J_{\theta-1,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{1,k} \end{bmatrix} \begin{bmatrix} f_{1,k,\theta} \\ f_{2,k,\theta} \\ \vdots \\ f_{\theta,k,\theta} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{\theta-1} \end{bmatrix} + \begin{bmatrix} -J_{0,k} f_{0,k,\theta} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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which is of the form:

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where

$$(F_{\theta,k})_{\beta,1} = f_{\beta,k,\theta} = \sum_{j=0}^{N-1} g_{j,\theta}^{(\beta)} e^{-i2\pi \frac{kj}{N}}, \qquad \beta = 1, 2, \dots, \theta$$

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- From [Cahill et al., 2002], the determinant of $M_{\theta,k}$ is given by

$$\det(M_{\theta,k}) = J_{1,k}\det(M_{\theta-1,k}) + \sum_{r=1}^{\theta-1}(-1)^{\theta-r}J_{\theta+1-r,k}(J_{0,k})^{\theta-r}\det(M_{r-1,k})$$

où

$$J_{p,k} = \begin{cases} e^{-i2\pi\frac{k}{N}} - 1, & \text{si } p = 0 \\ \dfrac{(\Delta t)^p}{p!}e^{-i2\pi\frac{k}{N}}, & \text{si } p > 0 \end{cases}$$

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- Goal : simplify $\det(M_{\theta,k})$

From [Pepin et al., 2020]

$$\det(M_{\theta,k}) = \frac{(\Delta t)^{\theta}}{\theta!} \sum_{\alpha=1}^{\theta} \left\langle \begin{matrix} \theta \\ \alpha - 1 \end{matrix} \right\rangle \left(e^{-i2\pi \frac{k}{N}}\right)^{\theta+1-\alpha}$$

with

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It is then possible to compute the numerical derivatives of g when N and θ are not simultaneously chosen as even integers. New

- The piecewise polynomials are built from truncated Taylor series:

$$\left[g^{(\beta)}\right]_j^{\theta}(t) = \sum_{p=0}^{\theta-\beta} \frac{(t-t_j)^p}{p!} g_{j,\theta}^{(p+\beta)}, \qquad t \in [t_j, t_{j+1}[$$

où $\beta = 0, 1, \ldots, \theta$.

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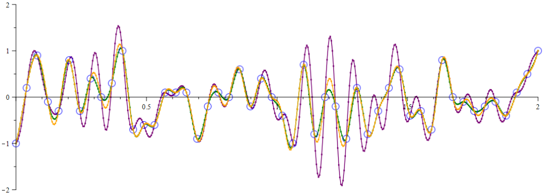


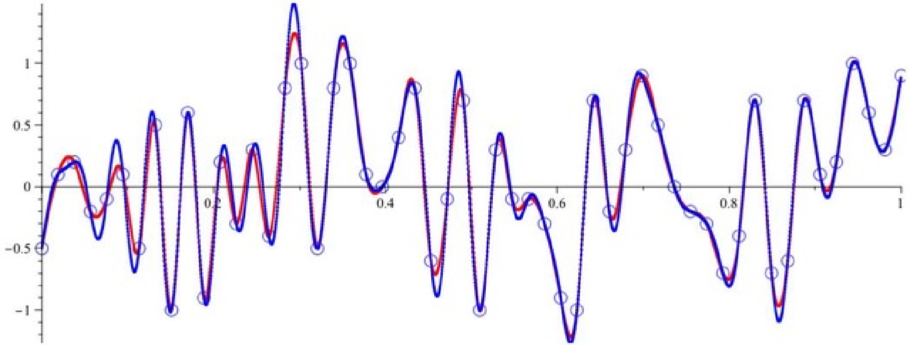
Green curve: $\theta = 3$ Purple curve: $\theta = 6$ Orange curve: $\theta = 9$

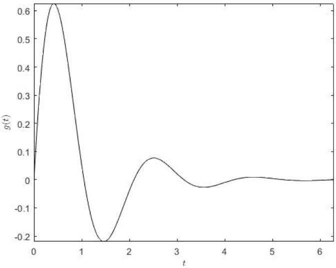
Table: $b_0 = g_N - g_0$ and $b_n = 0$, $n = 1, 2, \ldots, \theta - 1$.

Algorithms to approximate the boundary conditions

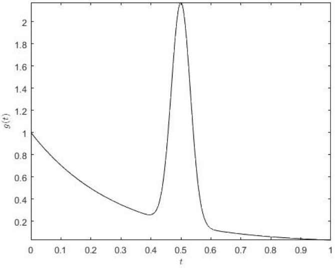
Methods based on:

$$\min_{B} \|g_\theta(t) - g_{\theta-1}(t)\|_2^2$$





(a) $g(t) = \sin(3t)e^{-t}$



(b) $g(t) = 2\exp\left(-500(t-\frac{1}{2})^2\right) + \exp\left(-\frac{7}{2}t\right)$

OM: Our method

Function: $g(t) = \sin(3t)e^{-t}$ on $[0, 2\pi]$

NS: Natural boundary conditions

NAK: Not-a-knot boundary conditions

N	Error	OM	NS	NAK
31	E_θ^{max}	2.43×10^{-03}	1.31×10^{-02}	3.59×10^{-03}
	E_θ^{avg}	9.44×10^{-05}	4.03×10^{-04}	1.28×10^{-04}
101	E_θ^{max}	1.07×10^{-04}	1.15×10^{-03}	3.92×10^{-05}
	E_θ^{avg}	1.19×10^{-06}	1.08×10^{-05}	5.65×10^{-07}
501	E_θ^{max}	9.79×10^{-07}	4.63×10^{-05}	6.65×10^{-08}
	E_θ^{avg}	2.21×10^{-09}	8.69×10^{-08}	5.18×10^{-10}

Table: $\theta = 3$

N	Error	OM
31	E_θ^{max}	5.51×10^{-04}
	E_θ^{avg}	1.80×10^{-05}
101	E_θ^{max}	6.08×10^{-07}
	E_θ^{avg}	6.20×10^{-09}
501	E_θ^{max}	7.15×10^{-11}
	E_θ^{avg}	1.54×10^{-13}

Table: $\theta = 5$

Function : $g(t) = \sin(3t)e^{-t}$ on $[0, 2\pi]$

N	Error	Our method
31	E_θ^{max}	4.11×10^{-06}
	E_θ^{avg}	9.37×10^{-08}
101	E_θ^{max}	1.13×10^{-11}
	E_θ^{avg}	7.80×10^{-14}
501	E_θ^{max}	4.67×10^{-19}
	E_θ^{avg}	6.53×10^{-22}

Table: $\theta = 11$

Function: $g(t) = 2\exp\left(-500(t-\frac{1}{2})^2\right) + \exp\left(-\frac{7}{2}t\right)$ on $[0,1]$

N	Error	Our method
31	E_θ^{max}	7.43×10^{-01}
	E_θ^{avg}	3.73×10^{-02}
101	E_θ^{max}	1.17×10^{-10}
	E_θ^{avg}	5.51×10^{-12}
501	E_θ^{max}	9.71×10^{-20}
	E_θ^{avg}	4.08×10^{-21}

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Table: $\theta = 11$

Solution: we need to add more interpolation nodes.

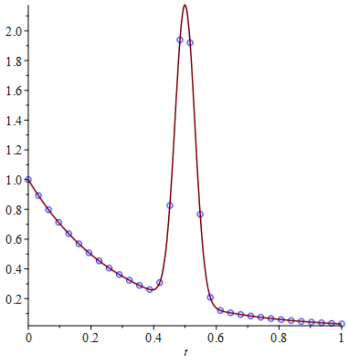
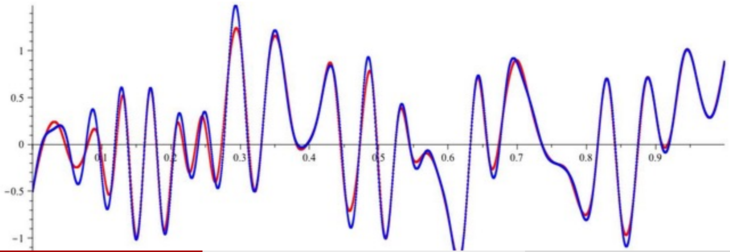


Figure: Distribution of $N+1=32$ interpolation nodes on the interval $[0,1]$ of the function $g(t)=2\exp\left(-500\left(t-\frac{1}{2}\right)^2\right)+\exp\left(-\frac{7}{2}t\right)$.

Conclusions

- New method to compute higher degree splines;
  - Continuity of the piecewise polynomials has been formally demonstrated;
  - The determinant of the matrix $M_{\theta,k}$ has been analyzed;
  - Algorithms for the approximation of the boundary conditions has been developed.

- Future projects: generalization to non equidistant nodes, generalization to higher dimension interpolation, in-depth study of the algorithms for the calculation of the boundary conditions, ...



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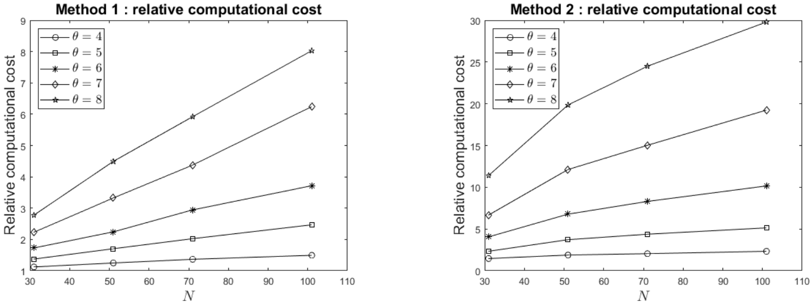

Figure: Relative computational cost for Algorithms 1 and 2 [Pepin et al., 2022].

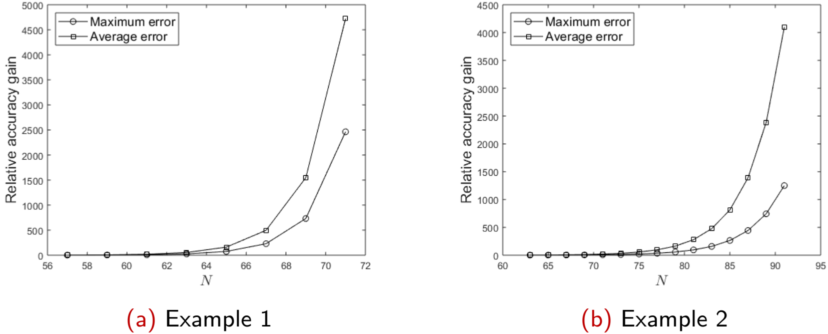

Figure: Relative accuracy gain ($\theta = 11$ vs $\theta = 3$) in terms of N.