Stabilité Asymptotique pour les Fluides avec Interfaces Diffuses

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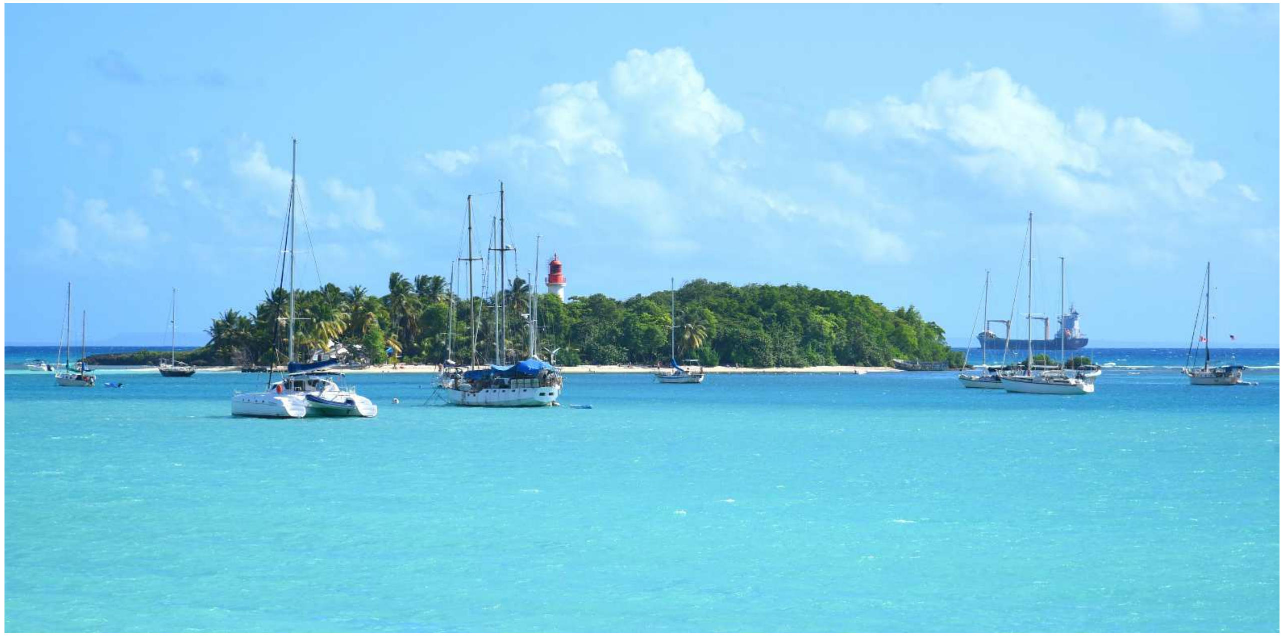

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Asymptotic Stability for Diffuse Interface Fluids

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2 Diffuse Interface Fluids

Diffuse Interface Models from Thermodynamics

- **Diffuse interface fluids from thermodynamics**

  Van der Waals (1891) Korteweg (1901) Dunn and Serrin (1985)

- **Cahn-Hilliard fluids from thermodynamics**

  Cahn and Hilliard (1958) (1959) Lowengrub and Truskinovsky (1997)
  Falk (1992) Verschueren (1999) Heida et al. (2012)

- **Ambiguity of thermodynamic derivations from kinetic derivation**

  Giovangigli (2020) (2021)

Diffuse Interface Fluids (1)

- **Van der Waals free energy $\boldsymbol{\mathcal{A} = \mathcal{A}^{\mathrm{cl}}(\rho, T) + \frac{1}{2}\varkappa|\nabla\rho|^2}$**

$$p = p^{\mathrm{cl}}(\rho, T) - \tfrac{1}{2}\varkappa|\boldsymbol{\nabla}\rho|^2 \qquad \mathcal{E} = \mathcal{E}^{\mathrm{cl}}(\rho, T) + \tfrac{1}{2}(\varkappa - T\partial_T\varkappa)|\boldsymbol{\nabla}\rho|^2$$

Gibbs relation $\quad T\,d\mathcal{S} = d\mathcal{E} - g\,d\rho - \varkappa\boldsymbol{\nabla}\rho\boldsymbol{\cdot}\,d\boldsymbol{\nabla}\rho$

- **Conservation equations**

$$\partial_t\rho + \boldsymbol{\nabla\cdot}(\rho\boldsymbol{v}) = 0$$

$$\partial_t(\rho\boldsymbol{v}) + \boldsymbol{\nabla\cdot}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) + \boldsymbol{\nabla\cdot}\boldsymbol{\mathcal{P}} = 0$$

$$\partial_t\big(\mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2\big) + \boldsymbol{\nabla\cdot}\Big(\boldsymbol{v}\big(\mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2\big)\Big) + \boldsymbol{\nabla\cdot}(\boldsymbol{\mathcal{Q}} + \boldsymbol{\mathcal{P}\cdot v}) = 0$$

Diffuse Interface Fluids (2)

- **Entropy balance**

$$\partial_t \mathcal{S} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{v}\mathcal{S}) + \boldsymbol{\nabla}\boldsymbol{\cdot}\Big(\frac{\boldsymbol{\mathcal{Q}}}{T} - \frac{\varkappa\rho\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}\boldsymbol{\nabla}\rho}{T}\Big)$$

$$= -\frac{1}{T}\Big(\boldsymbol{\mathcal{P}} - p\boldsymbol{I} - \varkappa\boldsymbol{\nabla}\rho\otimes\boldsymbol{\nabla}\rho + \rho\boldsymbol{\nabla}\boldsymbol{\cdot}(\varkappa\boldsymbol{\nabla}\rho)\,\boldsymbol{I}\Big)\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{v} - \Big(\boldsymbol{\mathcal{Q}} - \varkappa\rho\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}\boldsymbol{\nabla}\rho\Big)\boldsymbol{\cdot}\boldsymbol{\nabla}\Big(\frac{-1}{T}\Big)$$

- **Transport fluxes**

$$\boldsymbol{\mathcal{P}} = p\boldsymbol{I} + \varkappa\boldsymbol{\nabla}\rho\otimes\boldsymbol{\nabla}\rho - \rho\boldsymbol{\nabla}\boldsymbol{\cdot}(\varkappa\boldsymbol{\nabla}\rho)\boldsymbol{I} + \boldsymbol{\mathcal{P}}^{\mathrm{d}}$$

$$\boldsymbol{\mathcal{Q}} = \varkappa\rho\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}\,\boldsymbol{\nabla}\rho + \boldsymbol{\mathcal{Q}}^{\mathrm{d}}$$

$$\boldsymbol{\mathcal{P}}^{\mathrm{d}} = -\,\mathfrak{v}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}\,\boldsymbol{I} - \eta\Big(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^t - \tfrac{2}{d}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}\,\boldsymbol{I}\Big) \qquad \boldsymbol{\mathcal{Q}}^{\mathrm{d}} = -\lambda\boldsymbol{\nabla}T$$

Diffuse Interface Fluids (3)

- **Ambiguity of thermodynamics**

$$-\varkappa\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\ \boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}\left(\frac{-1}{T}\right)$$

- **Kinetic or molecular derivation**

  BBGKY hierarchy

  New Enskog type scaling of kinetic equations

  Simplification of pair distribution functions

  Taylor expansions of pair distribution functions

  Diffuse Interface and Cahn-Hilliard fluid equations

Diffuse Interface Fluids (4)

- **Thermodynamic stability**

Assume that $\mathsf{z}^{\mathrm{cl}} = (\rho, T)^t \;\mapsto\; \mathsf{u}^{\mathrm{cl}} = (\rho, \mathcal{E}^{\mathrm{cl}})^t$ is locally invertible then

$$\partial^2_{\mathsf{u}^{\mathrm{cl}}\mathsf{u}^{\mathrm{cl}}} \mathcal{S}^{\mathrm{cl}} \;\text{ negative definite } \iff \partial_T \mathcal{E}^{\mathrm{cl}} > 0 \text{ and } \partial_\rho p^{\mathrm{cl}} > 0$$

- **Assumptions on thermodynamics**

($\mathsf{H}_1^{\mathrm{cl}}$) *$\mathcal{E}^{\mathrm{cl}}$, p^{cl}, and $\mathcal{S}^{\mathrm{cl}}$ are C^γ functions of $\mathsf{z}^{\mathrm{cl}} = (\rho, T)^t$ over $\mathcal{O}_{\mathsf{z}^{\mathrm{cl}}}$*
$\mathcal{O}_{\mathsf{z}^{\mathrm{cl}}} \subset (0, \infty)^2$ simply connected nonempty open set.

($\mathsf{H}_2^{\mathrm{cl}}$) *Letting $\mathcal{G}^{\mathrm{cl}} = \mathcal{E}^{\mathrm{cl}} + p^{\mathrm{cl}} - T\mathcal{S}^{\mathrm{cl}} = \rho g^{\mathrm{cl}}$ then $T\, d\mathcal{S}^{\mathrm{cl}} = d\mathcal{E}^{\mathrm{cl}} - g^{\mathrm{cl}}\, d\rho$*

($\mathsf{H}_3^{\mathrm{cl}}$) *$\mathcal{O}_{\mathsf{z}^{\mathrm{cl}}}$ is increasing with T and $\partial_T \mathcal{E}^{\mathrm{cl}} > 0$*

Diffuse Interface Fluids (5)

- **Liquid-vapor equilibrium**

  $T_\mathrm{l} = T_\mathrm{g} \quad p_\mathrm{l} = p_\mathrm{g} \quad g_\mathrm{l} = g_\mathrm{g} \qquad g = a + p/\rho$ Gibbs function

- **Equilibrium liquid-vapor interfaces**

  One dimensional steady interface $\qquad z$ the normal variable

  Extremalizing entropy with given energy and mass

  Isotherm interface $T(z) = T_\mathrm{l} = T_\mathrm{g}$

  $\frac{1}{2}\varkappa(\mathrm{d}\rho/\mathrm{d}z)^2 = \mathcal{A} - \mathcal{A}_\mathrm{g} - g_\mathrm{g}(\rho - \rho_\mathrm{g}) \approx \overline{\mathcal{A}}(\rho - \rho_\mathrm{l})^2(\rho - \rho_\mathrm{g})^2$

$$\rho(z) = \tfrac{1}{2}(\rho_\mathrm{l} + \rho_\mathrm{g}) + \tfrac{1}{2}(\rho_\mathrm{l} - \rho_\mathrm{g})\tanh(z/2\overline{z})$$

  Epaisseur de l'interface $\quad \overline{z} = (\overline{\varkappa}/2\overline{\mathcal{A}})^{1/2}/(\rho_\mathrm{l} - \rho_\mathrm{g})$

3 Augmented System

Augmented Systems for Diffuse Interface Models

- **Augmented systems**

  Gavrilyuk and Gouin (1999) Benzoni et al. (2005) (2006) (2007)
  Bresch et al. (2019) (2000) Kotschote (2012)

- **Two velocity hydrodynamics**

  Bresch et al. (2008) (2015) (2015)

- **Symmetrization of the augmented system**

  Gavrilyuk and Gouin (1999) (2000)

Augmented system (1)

- **Extra unknown** $\boldsymbol{w} = \boldsymbol{\nabla}\rho$

$$\partial_t \boldsymbol{w} + \sum_{i\in\mathcal{D}} \partial_i\big(\boldsymbol{w}\, v_i + \rho \boldsymbol{\nabla} v_i\big) = 0 \qquad \mathcal{D} = \{\, 1, \ldots, d\,\}$$

- **Augmented unknowns**

$$\mathsf{u} = \big(\rho, \boldsymbol{w}, \rho\boldsymbol{v}, \mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2\big)^t \qquad \mathsf{z} = \big(\rho, \boldsymbol{w}, \boldsymbol{v}, T\big)^t$$

- **New thermodynamic functions**

$$\mathcal{E} = \mathcal{E}^{\mathrm{cl}} + \tfrac{1}{2}(\varkappa - T\partial_T\varkappa)|\boldsymbol{w}|^2 \qquad \mathcal{S} = \mathcal{S}^{\mathrm{cl}} - \tfrac{1}{2}\partial_T\varkappa|\boldsymbol{w}|^2$$

$$p = p^{\mathrm{cl}} - \tfrac{1}{2}\varkappa|\boldsymbol{w}|^2 \qquad g = g^{\mathrm{cl}}$$

Augmented system (2)

- **Thermodynamic functions**

(H₁) *$\mathcal{E}$, p, $\mathcal{S}$ are C^γ functions of $\mathsf{z} \in \mathcal{O}_\mathsf{z} \subset (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\times(0,\infty)$ open set*

$\varkappa = \varkappa(T)$ is a $C^{\gamma+1}$ function of temperature T over $\mathcal{O}_\mathsf{z}$

If $(\rho,T)^t \in \mathcal{O}_{\mathsf{z}^{\mathrm{cl}}}$, $(\rho,0,0,T)^t \in \mathcal{O}_\mathsf{z}$. If $(\rho,\boldsymbol{w},\boldsymbol{v},T)^t \in \mathcal{O}_\mathsf{z}$, $(\rho,T)^t \in \mathcal{O}_{\mathsf{z}^{\mathrm{cl}}}$

(H₂) *Letting $\mathcal{G} = \mathcal{E} + p - T\mathcal{S} = \rho g$ we have $T\,d\mathcal{S} = d\mathcal{E} - g\,d\rho - \varkappa\boldsymbol{w}\boldsymbol{\cdot}d\boldsymbol{w}$*

(H₃) *The open set $\mathcal{O}_\mathsf{z}$ is increasing with temperature T and $\partial_T\mathcal{E} > 0$*

(H₄) *The capillarity coefficient is positive $\varkappa > 0$ over $\mathcal{O}_\mathsf{z}$*

(H₅) *The coefficients $\mathfrak{v}$, η and λ are C^γ functions over $\mathcal{O}_\mathsf{z}$*

We have $\eta > 0$, $\lambda > 0$, $\mathfrak{v} \geq 0$, and $\mathfrak{v} + \eta(1-\frac{2}{d}) > 0$

Augmented system (3)

Lemma 1. *Assuming* (H_1)-(H_2) *and that* $\mathsf{z} \mapsto \mathsf{u}$ *is locally invertible then*

$$\partial^2_{\mathsf{uu}}\mathcal{S} \text{ negative definite} \iff \partial_T\mathcal{E} > 0 \quad \partial_\rho p > 0 \quad \text{and} \quad \varkappa > 0$$

Lemma 2. *Assuming* (H_1)-(H_3) *then the map* $\mathsf{z} \mapsto \mathsf{u}$ *is a* C^γ *diffeomorphism from the open set* $\mathcal{O}_\mathsf{z}$ *onto an open set* $\mathcal{O}_\mathsf{u}$*.*

Lemma 3. *Assuming* (H_1) *and given* $\delta > 0$ *there exists a* $C^{\gamma-1}$ *function* m *such that* $\mathsf{m} \geq 0$

$$\mathsf{m} + \partial_\rho p/\rho T > 0$$

and $\mathsf{m} = 0$ *if* $\partial_\rho p/\rho T \geq \delta$*.*

Augmented system (4)

- **Augmented entropic variable**

$$\sigma = -\mathcal{S} = -\mathcal{S}^{\mathrm{cl}} + \tfrac{1}{2}\partial_T \varkappa\, |\boldsymbol{w}|^2 \qquad \mathsf{v} = (\partial_{\mathsf{u}}\sigma)^t = \frac{1}{T}\Big(g - \tfrac{1}{2}|\boldsymbol{v}|^2, \varkappa\,\boldsymbol{w}, \boldsymbol{v}, -1\Big)^t$$

- **Stable points**

$$\mathcal{O}_{\mathsf{z}}^{\mathrm{st}} = \{\ \mathsf{z} \in \mathcal{O}_{\mathsf{z}}\ \big|\ \partial_\rho p > 0\ \}$$

$\mathsf{u} \mapsto \mathsf{v}$ locally invertible around stable points with $\partial_\rho p > 0$

- **Legendre transform $\mathcal{L}$ of entropy**

$$\mathcal{L} = \langle \mathsf{u}, \mathsf{v} \rangle - \sigma = \frac{1}{T}\big(p + \varkappa|\boldsymbol{w}|^2\big) \qquad \partial_{\mathsf{u}}\sigma = \mathsf{v}^t \qquad \partial_{\mathsf{v}}\mathcal{L} = \mathsf{u}^t$$

- **Convective fluxes**

$$\mathsf{F}_i = \big(\partial_{\mathsf{v}}(\mathcal{L}v_i)\big)^t \qquad \mathcal{L}_i = \mathcal{L}v_i$$

Augmented system (5)

- **New augmented form**

$$\partial_t \mathsf{u} + \sum_{i\in\mathcal{D}} \partial_i\big(\mathsf{F}_i + \mathsf{F}_i^{\mathrm{d}} + \mathsf{F}_i^{\mathrm{c}}\big) = 0$$

- **New augmented fluxes inthe ith direction**

$$\mathsf{F}_i = \Big(\rho v_i, \boldsymbol{w} v_i, \rho \boldsymbol{v} v_i + (p + \varkappa|\boldsymbol{w}|^2)\mathfrak{b}_i, (\mathcal{E} + p + \varkappa|\boldsymbol{w}|^2)v_i\Big)^t$$

$$\mathsf{F}_i^{\mathrm{d}} = \Big(0, 0_{d,1}, \boldsymbol{\mathcal{P}}_i^{\mathrm{d}}, \mathcal{Q}_i^{\mathrm{d}} + \sum_{j\in\mathcal{D}} \mathcal{P}_{ij}^{\mathrm{d}} v_j\Big)^t \qquad \boldsymbol{\mathcal{P}}_i^{\mathrm{d}} = (\mathcal{P}_{1i}^{\mathrm{d}}, \ldots, \mathcal{P}_{di}^{\mathrm{d}})^t$$

$$\mathsf{F}_i^{\mathrm{c}} = \Big(0, \rho\boldsymbol{\nabla} v_i, -\rho\boldsymbol{\nabla}(\varkappa w_i), \rho\varkappa\boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla} v_i - \rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}(\varkappa w_i)\Big)^t$$

- **Equivalence of both formulations**

  Rely on calculus identities

Augmented system (6)

- **Convective, dissipative and capillary matrices**

$$\mathsf{A}_i = \partial_{\mathsf{u}} \mathsf{F}_i \qquad \mathsf{F}_i^{\mathrm{d}} = -\sum_{j\in\mathcal{D}} \mathsf{B}_{ij}^{\mathrm{d}} \partial_j \mathsf{u} \qquad \mathsf{F}_i^{\mathrm{c}} = -\sum_{j\in\mathcal{D}} \mathsf{B}_{ij}^{\mathrm{c}} \partial_j \mathsf{u}, \qquad i \in \mathcal{D}$$

- **Quasilinear form**

$$\partial_t \mathsf{u} + \sum_{i\in\mathcal{D}} \mathsf{A}_i(\mathsf{u}) \partial_i \mathsf{u} - \sum_{i,j\in\mathcal{D}} \partial_i \big(\mathsf{B}_{ij}^{\mathrm{d}}(\mathsf{u}) \partial_j \mathsf{u} \big) - \sum_{i,j\in\mathcal{D}} \partial_i \big(\mathsf{B}_{ij}^{\mathrm{c}}(\mathsf{u}) \partial_j \mathsf{u} \big) = 0$$

A_i, $\mathsf{B}_{ij}^{\mathrm{d}}$, and $\mathsf{B}_{ij}^{\mathrm{c}}$, for $i, j \in \mathcal{D}$, have at least regularity $C^{\gamma-1}$ over $\mathcal{O}_{\mathsf{u}}$

- **Symmetrization**

Structure of the system of equations plus existence results

Symmetrized Augmented System (1)

- **Entropic symmetrization for stable points $\mathbf{u} = \mathbf{u}(\mathbf{v})$**

$$\widetilde{\mathsf{A}}_0(\mathsf{v})\partial_t \mathsf{v} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}_i(\mathsf{v})\partial_i \mathsf{v} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{v})\partial_j \mathsf{v}\big) - \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{v})\partial_j \mathsf{v}\big) = 0$$

$$\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{v}}\mathsf{u} \quad \widetilde{\mathsf{A}}_i = \mathsf{A}_i\partial_{\mathsf{v}}\mathsf{u} \quad \widetilde{\mathsf{B}}^{\mathrm{d}}_{ij} = \mathsf{B}^{\mathrm{d}}_{ij}\partial_{\mathsf{v}}\mathsf{u} \quad \widetilde{\mathsf{B}}^{\mathrm{c}}_{ij} = \mathsf{B}^{\mathrm{c}}_{ij}\partial_{\mathsf{v}}\mathsf{u} \quad \det \widetilde{\mathsf{A}}_0 = \frac{\rho^2 T^5}{\varkappa}\,\frac{\partial_T \mathcal{E}}{\partial_\rho p}$$

- **Structure of entropic symmetrized system**

$\widetilde{\mathsf{A}}_0$ symmetric positive definite for stable points $\qquad \widetilde{\mathsf{A}}_i$ symmetric for $i \in \mathcal{D}$

$$(\widetilde{\mathsf{B}}^{\mathrm{d}}_{ij})^t = \widetilde{\mathsf{B}}^{\mathrm{d}}_{ji} \qquad \sum_{i,j\in\mathcal{D}} \xi_i\xi_j \widetilde{\mathsf{B}}^{\mathrm{d}}_{ij} \text{ positive semi definite} \qquad (\widetilde{\mathsf{B}}^{\mathrm{c}}_{ij})^t = -\widetilde{\mathsf{B}}^{\mathrm{c}}_{ji}$$

The map $\mathsf{u} \mapsto \mathsf{v} = (\partial_{\mathsf{u}}\sigma)^t = \frac{1}{T}\Big(g - \frac{1}{2}|\boldsymbol{v}|^2, \varkappa\,\boldsymbol{w}, \boldsymbol{v}, -1\Big)^t$ is generally not globally invertible

Symmetrized Augmented System (2)

- **Normal variable**

$\mathsf{w} = \big(\rho, \boldsymbol{w}, \boldsymbol{v}, T\big)^t \quad \mathsf{w} = (\mathsf{w}_{\mathrm{I}}, \mathsf{w}_{\mathrm{II}})^t \quad \mathsf{w}_{\mathrm{I}} = (\rho, \boldsymbol{w})^t \quad \mathsf{w}_{\mathrm{II}} = (\boldsymbol{v}, T)^t$

$\mathbb{R}^{\mathsf{n}} = \mathbb{R}^{\mathsf{n}_{\mathrm{I}}} \times \mathbb{R}^{\mathsf{n}_{\mathrm{II}}} \quad \mathsf{n} = \mathsf{n}_{\mathrm{I}} + \mathsf{n}_{\mathrm{II}} \quad \mathsf{n}_{\mathrm{I}} = \mathsf{n}_{\mathrm{II}} = d + 1$

$\mathsf{w}_{\mathrm{I}} = (\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{I}''})^t \quad \mathsf{w}_{\mathrm{I}'} = \rho \quad \mathsf{w}_{\mathrm{I}''} = \boldsymbol{w} \quad \boldsymbol{\nabla} \mathsf{w}_{\mathrm{I}'} = \mathsf{w}_{\mathrm{I}''} \quad \mathsf{w}_{\mathrm{r}} = (\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{II}})^t$

$\mathsf{u} \to \mathsf{w}$ diffeomorphism from $\mathcal{O}_{\mathsf{u}}$ onto $\mathcal{O}_{\mathsf{w}} = \mathcal{O}_{\mathsf{z}}$ since $\mathsf{w} = \mathsf{z}$

- **Normal form**

$\mathsf{u} = \mathsf{u}(\mathsf{w})$ and multiplication on the left by $(\partial_{\mathsf{w}} \mathsf{v})^t$

Add $\big(\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v})\big) \times \mathsf{m}$ to the first equation Non conservative form

$\overline{\mathsf{A}}_0 = (\partial_{\mathsf{w}} \mathsf{v})^t \partial_{\mathsf{w}} \mathsf{u} + \mathsf{m}\, \mathsf{e}_1 \otimes \mathsf{e}_1 \qquad \overline{\mathsf{A}}_i = (\partial_{\mathsf{w}} \mathsf{v})^t \partial_{\mathsf{w}} \mathsf{F}_i + \mathsf{m} v_i\, \mathsf{e}_1 \otimes \mathsf{e}_1 \quad i \in \mathcal{D}$

$$\overline{\mathsf{A}}_0(\mathsf{w}) \partial_t \mathsf{w} + \sum_{i \in \mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w}) \partial_i \mathsf{w} - \sum_{i,j \in \mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w}) \partial_i \partial_j \mathsf{w} - \sum_{i,j \in \mathcal{D}} \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w}) \partial_i \partial_j \mathsf{w} = \mathsf{h}(\mathsf{w}, \boldsymbol{\nabla} \mathsf{w})$$

Symmetrized Augmented System (3)

- **Normal form**

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t \mathsf{w} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w})\partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j \mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j \mathsf{w} = \mathsf{h}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$$

- **Properties of the normal form**

$\overline{\mathsf{A}}_0 = \mathrm{diag}(\overline{\mathsf{A}}_0^{\mathrm{I,I}}, \overline{\mathsf{A}}_0^{\mathrm{II,II}})$ symmetric positive definite $\qquad \overline{\mathsf{A}}_i$ symmetic for $i \in \mathcal{D}$

$(\overline{\mathsf{B}}^{\mathrm{d}}_{ij})^t = \overline{\mathsf{B}}^{\mathrm{d}}_{ji} \qquad \overline{\mathsf{B}}^{\mathrm{d}}_{ij} = \mathrm{diag}(0, \overline{\mathsf{B}}^{\mathrm{d\,II,II}}_{ij}) \qquad \overline{\mathsf{B}}^{\mathrm{d\,II,II}} = \sum_{i,j\in\mathcal{D}} \xi_i\xi_j \overline{\mathsf{B}}^{\mathrm{d\,II,II}}_{ij}$ positive definite

$(\overline{\mathsf{B}}^{\mathrm{c}}_{ij})^t = -\overline{\mathsf{B}}^{\mathrm{c}}_{ji} \qquad \overline{\mathsf{B}}^{\mathrm{c\,I,I}}_{ij} = 0 \qquad \overline{\mathsf{B}}^{\mathrm{c\,I,II}}_{ij},\ \overline{\mathsf{B}}^{\mathrm{c\,II,I}}_{ij},\ \overline{\mathsf{A}}_0^{\mathrm{II,II}}$ depend on $\mathsf{w}_{\mathrm{r}} = (\mathsf{w}_{\mathrm{I'}}, \mathsf{w}_{\mathrm{II}})^t$

- **Right hand side**

$$\mathsf{h} = (\mathsf{h}_{\mathrm{I}}, \mathsf{h}_{\mathrm{II}})^t \qquad \mathsf{h}_{\mathrm{I}} = \Big(-\mathsf{m}\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}, -\frac{\varkappa}{T}\sum_{i\in\mathcal{D}} w_i \boldsymbol{\nabla} v_i\Big)^t \qquad \mathsf{h}_{\mathrm{II}} = \mathsf{h}_{\mathrm{II}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$$

Symmetrized Augmented System (4)

- **Gradient constraint for nonlinear equations**

  Natural equation for $\boldsymbol{w} - \boldsymbol{\nabla}\rho$

$$\partial_t(\boldsymbol{w} - \boldsymbol{\nabla}\rho) + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}(\boldsymbol{w} - \boldsymbol{\nabla}\rho) + (\boldsymbol{w} - \boldsymbol{\nabla}\rho)\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} + (\boldsymbol{\nabla}\boldsymbol{v})^t\boldsymbol{\cdot}(\boldsymbol{w} - \boldsymbol{\nabla}\rho) = 0$$

  If w is smooth enough, $\boldsymbol{w}_0 - \boldsymbol{\nabla}\rho_0 = 0$ and $\boldsymbol{w}^\star = 0$ then $\boldsymbol{w} - \boldsymbol{\nabla}\rho = 0$

- **Linearized equation with gradient constraint**

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} =$$

$$\Big(-\mathsf{m}\,\rho\,\boldsymbol{\nabla}\boldsymbol{\cdot}\widetilde{\boldsymbol{v}}, -\sum_{i\in\mathcal{D}}\frac{\varkappa}{T}\widetilde{w}_i\boldsymbol{\nabla}v_i, \mathsf{h}_{\mathrm{II}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})\Big)^t$$

Symmetrized Augmented System (5)

- **Linearized equation with gradient constraint**

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i'(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}}$$

$$+\overline{\mathsf{L}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}_{\mathrm{II}})\widetilde{\mathsf{w}} = \mathsf{h}'(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}) = \big(0, \mathsf{h}_{\mathrm{II}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})\big)^t$$

$$\overline{\mathsf{A}}_i'(\mathsf{w}) = \overline{\mathsf{A}}_i(\mathsf{w}) + \mathsf{m}\rho\mathsf{e}_1\otimes\mathsf{e}_{d+1+i} \qquad \overline{\mathsf{L}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}_{\mathrm{II}}) = \sum_{i\in\mathcal{D}} \frac{\varkappa}{T}(0, \boldsymbol{\nabla}v_i, 0_{1,\mathsf{n}_{\mathrm{I}}}, 0)^t\otimes\mathsf{e}_{i+1}$$

- **Gradient constraint for linearized equations**

Natural equation for $\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho}$

$$\partial_t(\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho}) + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}(\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho}) + (\boldsymbol{w} - \boldsymbol{\nabla}\rho)\,\boldsymbol{\nabla}\boldsymbol{\cdot}\widetilde{\boldsymbol{v}} + \boldsymbol{\nabla}\boldsymbol{v}^t\boldsymbol{\cdot}(\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho}) = 0$$

If w and $\widetilde{\mathsf{w}}$ are regular, $\boldsymbol{w} - \boldsymbol{\nabla}\rho = 0$, $\widetilde{\boldsymbol{w}}_0 - \boldsymbol{\nabla}\widetilde{\rho}_0 = 0$, $\widetilde{\boldsymbol{w}}^\star = 0$ then $\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho} = 0$

4 Linearized Estimates and Local Existence of Solutions

Linearized Equations (1)

- **Linearized equations**

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i'(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} + \overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})\widetilde{\mathsf{w}} = \mathsf{f}+\mathsf{g}$$

- **Assumptions on the coefficients**

$\overline{\mathsf{A}}_0 = \mathrm{diag}(\overline{\mathsf{A}}_0^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})$ symmetric positive definite block diagonal

$\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}$ are symmetric, $(\overline{\mathsf{B}}_{ij}^{\mathrm{d}})^t = \overline{\mathsf{B}}_{ji}^{\mathrm{d}}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{d}} = \mathrm{diag}(0, \overline{\mathsf{B}}_{ij}^{\mathrm{d}\,\mathrm{II},\mathrm{II}})$

$\overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}} = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}\xi_i\xi_j$ is positive definite for $\boldsymbol{\xi}\in\Sigma^{d-1}$

$(\overline{\mathsf{B}}_{ij}^{\mathrm{c}})^t = -\overline{\mathsf{B}}_{ji}^{\mathrm{c}}$ $\quad \overline{\mathsf{B}}_{ij}^{\mathrm{c},\mathrm{I},\mathrm{I}} = 0$ $\quad \overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{c},\mathrm{I},\mathrm{II}}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{c},\mathrm{II},\mathrm{I}}$ only depend on $\mathsf{w}_{\mathrm{r}} = (\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{II}})^t$

$\overline{\mathsf{L}} = \mathrm{diag}(\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}})$ $\quad \overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}} = \mathfrak{L}^{\mathrm{I},\mathrm{I}}(\mathsf{w})\boldsymbol{\nabla}\mathsf{w}_{\mathrm{r}}$ $\quad \overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}} = \mathfrak{L}^{\mathrm{II},\mathrm{II}}(\mathsf{w})\boldsymbol{\nabla}\mathsf{w}_{\mathrm{r}}$

$\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i'$, $\overline{\mathsf{B}}_{ij}^{\mathrm{d}}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{c}}$, $\mathfrak{L}^{\mathrm{I},\mathrm{I}}$, $\mathfrak{L}^{\mathrm{II},\mathrm{II}}$ are C^{l+2} over $\mathcal{O}_{\mathsf{w}}$ $\qquad \overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})\,\widetilde{\mathsf{w}}^\star = 0$

Linearized Equations (2)

- **Assumptions on w**

  $d \geq 1 \quad l \geq l_0 + 2$ where $l_0 = [d/2] + 1 \quad 1 \leq l' \leq l$

  w given function of $(t, \boldsymbol{x})$ over $[0, \bar{\tau}] \times \mathbb{R}^d$ with $\bar{\tau} > 0$

$$\begin{cases} \mathsf{w}_{\mathrm{I}} - \mathsf{w}^\star_{\mathrm{I}} \in C^0\big([0,\bar{\tau}], H^l\big) \cap C^1\big([0,\bar{\tau}], H^{l-2}\big) \\ \mathsf{w}_{\mathrm{II}} - \mathsf{w}^\star_{\mathrm{II}} \in C^0\big([0,\bar{\tau}], H^l\big) \cap C^1\big([0,\bar{\tau}], H^{l-2}\big) \cap L^2\big((0,\bar{\tau}), H^{l+1}\big) \end{cases}$$

  $\mathcal{O}_0 \subset \overline{\mathcal{O}}_0 \subset \mathcal{O}_{\mathsf{w}},\ 0 < a_1 < \mathrm{dist}(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathsf{w}}),\ \ \mathcal{O}_1 = \{\, \mathsf{w} \in \mathcal{O}_{\mathsf{w}};\ \mathrm{dist}(\mathsf{w}, \overline{\mathcal{O}}_0) < a_1 \,\}$

  $\mathsf{w}_0(\boldsymbol{x}) = \mathsf{w}(0, \boldsymbol{x}) \in \mathcal{O}_0,\ \mathsf{w}(t, \boldsymbol{x}) \in \mathcal{O}_1,\ (t, \boldsymbol{x}) \in [0, \bar{\tau}] \times \mathbb{R}^d$

- **Assumptions on f and g**

  f and g given functions of $(t, \boldsymbol{x})$ over $[0, \bar{\tau}] \times \mathbb{R}^d \qquad 1 \leq l' \leq l$

  $\mathsf{f} \in C^0\big([0,\bar{\tau}], H^{l'-1}\big) \cap L^1\big((0,\bar{\tau}), H^{l'}\big) \qquad \mathsf{g} \in C^0\big([0,\bar{\tau}], H^{l'-1}\big) \quad \mathsf{g}_{\mathrm{I}} = 0$

Linearized Equations (3)

- **Assumptions on $\widetilde{\mathsf{w}}$**

$$\widetilde{\mathsf{w}}_{\mathrm{I}} - \widetilde{\mathsf{w}}^{\star}_{\mathrm{I}} \in C^0\big([0,\bar\tau], H^{l'}\big) \cap C^1\big([0,\bar\tau], H^{l'-2}\big),$$

$$\widetilde{\mathsf{w}}_{\mathrm{II}} - \widetilde{\mathsf{w}}^{\star}_{\mathrm{II}} \in C^0\big([0,\bar\tau], H^{l'}\big) \cap C^1\big([0,\bar\tau], H^{l'-2}\big) \cap L^2\big((0,\bar\tau), H^{l'+1}\big),$$

- **Bounding quantities**

$$M^2 = \sup_{0\le\tau\le\bar\tau} |\mathsf{w}(\tau) - \mathsf{w}^{\star}|_l^2, \qquad M_{\mathrm{t}}^2 = \int_0^{\bar\tau} |\partial_t \mathsf{w}(\tau)|_{l-2}^2\, d\tau, \qquad M_{\mathrm{r}}^2 = \int_0^{\bar\tau} |\nabla \mathsf{w}_{\mathrm{r}}(\tau)|_l^2\, d\tau$$

- **Linearized estimates for $1 \le l' \le l$**

There exists constants $\mathsf{c}_1(\mathcal{O}_1) \ge 1$ and $\mathsf{c}_2(\mathcal{O}_1, M) \ge 1$ increasing with M with

$$\sup_{0\le\tau\le t} |\widetilde{\mathsf{w}}(\tau) - \widetilde{\mathsf{w}}^{\star}|_{l'}^2 + \int_0^t |\widetilde{\mathsf{w}}_{\mathrm{II}}(\tau) - \widetilde{\mathsf{w}}^{\star}_{\mathrm{II}}|_{l'+1}^2\, d\tau \le \mathsf{c}_1^2 \exp\Big(\mathsf{c}_2\big(t + M_{\mathrm{t}}\sqrt{t} + M_{\mathrm{r}}\sqrt{t}\,\big)\Big) \times$$

$$\Big(|\widetilde{\mathsf{w}}_0 - \widetilde{\mathsf{w}}^{\star}|_{l'}^2 + \mathsf{c}_2 \Big\{ \int_0^t |\mathsf{f}|_{l'}\, d\tau \Big\}^2 + \mathsf{c}_2 \int_0^t |\mathsf{g}_{\mathrm{II}}|_{l'-1}^2\, d\tau \Big)$$

Linearized Equations (4)

- **Sketch of the proof for the linearized estimates**

  Notation $\delta\widetilde{\mathsf{w}} = \widetilde{\mathsf{w}} - \widetilde{\mathsf{w}}^\star$ and $E_k^2(\phi) = \sum_{0\le|\alpha|\le k} \frac{|\alpha|!}{\alpha!} \int_{\mathbb{R}^d} \langle\, \overline{\mathsf{A}}_0(\mathsf{w})\partial^\alpha\phi, \partial^\alpha\phi\rangle\, d\boldsymbol{x}$

  Use of Gronwall Lemma and the inequality ($\delta(\mathcal{O}_1) \le 1$ small constant)

$$\partial_t E_{l'}^2(\delta\widetilde{\mathsf{w}}) + \delta_1|\delta\widetilde{\mathsf{w}}_{\rm II}|_{l'+1}^2 \le \mathsf{c}_2\big(1 + |\partial_t \mathsf{w}|_{l-2} + |\nabla \mathsf{w}_{\rm r}|_l\big) E_{l'}^2(\delta\widetilde{\mathsf{w}})$$

$$+ \mathsf{c}_2|\mathsf{f}|_{l'} E_{l'}(\delta\widetilde{\mathsf{w}}) + \mathsf{c}_2|\mathsf{g}_{\rm II}|_{l'-1}^2$$

- **Zeroth order inequality $\boldsymbol{k = 0}$**

  ⋆ Multiply the equation by $\delta\widetilde{\mathsf{w}}$ and integrate over $\mathbb{R}^d$

  ⋆ Time derivative terms estimated with the symmetry of $\overline{\mathsf{A}}_0$

$$\langle \delta\widetilde{\mathsf{w}}, \overline{\mathsf{A}}_0(\mathsf{w})\partial_t\delta\widetilde{\mathsf{w}}\rangle = \tfrac{1}{2}\partial_t\langle\delta\widetilde{\mathsf{w}}, \overline{\mathsf{A}}_0(\mathsf{w})\delta\widetilde{\mathsf{w}}\rangle - \tfrac{1}{2}\langle\delta\widetilde{\mathsf{w}}, \partial_t\overline{\mathsf{A}}_0(\mathsf{w})\delta\widetilde{\mathsf{w}}\rangle,$$

  $\partial_t\overline{\mathsf{A}}_0(\mathsf{w}) = \partial_{\mathsf{w}}\overline{\mathsf{A}}_0\,\partial_t\mathsf{w}$ is estimated with $|\partial_t\overline{\mathsf{A}}_0|_{L^\infty} \le \mathsf{c}_0|\partial_t\overline{\mathsf{A}}_0|_{l-2} \le \mathsf{c}_1|\partial_t\mathsf{w}|_{l-2}$

Linearized Equations (5)

- **Zeroth order inequality $k = 0$ (continued)**
  - ⋆ The products $\langle \delta\widetilde{\mathsf{w}}, \overline{\mathsf{A}}_i'(\mathsf{w})\partial_i\delta\widetilde{\mathsf{w}}\rangle$ are evaluated by blocks

    Symmetry for the $(\textsc{i},\textsc{i})$ terms, direct estimates for $(\textsc{i},\textsc{ii})$ and $(\textsc{ii},\textsc{ii})$ terms

    The $(\textsc{ii},\textsc{i})$ terms are integrated by parts, $|\overline{\mathsf{A}}_i|_{L^\infty} \leq \mathsf{c}_1$ and $|\partial_i\overline{\mathsf{A}}_i|_{L^\infty} \leq \mathsf{c}_2$
  - ⋆ Dissipative terms integrated by parts, $|\partial_i\overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})|_{L^\infty} \leq \mathsf{c}_2$, Garding inequality

$$\delta_1|\phi_{\textsc{ii}}|_1^2 \leq \sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^d} \langle \overline{\mathsf{B}}^{\mathrm{d}\,\textsc{ii},\textsc{ii}}_{ij}(\mathsf{w})\partial_j\phi_{\textsc{ii}}, \partial_i\phi_{\textsc{ii}}\rangle\, d\boldsymbol{x} + \mathsf{c}_2|\phi_{\textsc{ii}}|_0^2 \qquad \phi_{\textsc{ii}} \in H^1(\mathbb{R}^d)$$

  - ⋆ Antisymmetric terms integrated by parts and the first sum vanishes

$$-\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^d} \langle \delta\widetilde{\mathsf{w}}, \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x} = \sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^d} \langle \partial_i\delta\widetilde{\mathsf{w}}, \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x}$$

$$+ \sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^d} \langle \delta\widetilde{\mathsf{w}}, \partial_i\overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x}.$$

Linearized Equations (6)

- **Zeroth order inequality $k = 0$ (continued)**
  - ⋆ Block evaluation of the terms $\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^d}\langle\delta\widetilde{\mathsf{w}},\partial_i\overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x}$

    The terms (I, I), (I, II), (II, II) easily estimated, (II, I) terms integrated by parts and use of Use of $|\partial_i\partial_j\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}| \leq \mathsf{c}_2$ since $l \geq l_0 + 2$

$$\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^d}\langle\delta\widetilde{\mathsf{w}}_{\mathrm{II}},\partial_i\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\,d\boldsymbol{x} = -\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^d}\langle\partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{II}},\partial_i\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w})\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\,d\boldsymbol{x}$$
$$-\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^d}\langle\delta\widetilde{\mathsf{w}}_{\mathrm{II}},\partial_i\partial_j\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w})\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\,d\boldsymbol{x}.$$

  - ⋆ Zeroth order terms $\int_{\mathbb{R}^d}\langle\delta\widetilde{\mathsf{w}},\overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x} \leq \mathsf{c}_2|\delta\widetilde{\mathsf{w}}|_0^2$ and right hand side terms $\int_{\mathbb{R}^d}\langle\delta\widetilde{\mathsf{w}},\mathsf{f}\rangle\,d\boldsymbol{x} \leq \mathsf{c}_1|\delta\widetilde{\mathsf{w}}|_0|\mathsf{f}|_0$ and $\int_{\mathbb{R}^d}\langle\delta\widetilde{\mathsf{w}},\mathsf{g}\rangle\,d\boldsymbol{x} \leq |\delta\widetilde{\mathsf{w}}|_0|\mathsf{g}|_0$

$$\partial_t E_0^2(\delta\widetilde{\mathsf{w}}) + \delta_1|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1^2 \leq \mathsf{c}_1|\mathsf{f}|_0|\delta\widetilde{\mathsf{w}}|_0 + \mathsf{c}_1|\mathsf{g}_{\mathrm{II}}|_0^2 + \mathsf{c}_2(1+|\partial_t\mathsf{w}|_{l-2})E_0^2(\delta\widetilde{\mathsf{w}}).$$

Linearized Equations (7)

- **The l'th order inequality**

  ⋆ The l'th order inequality obtained from

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\partial^\alpha\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}'_i(\mathsf{w})\partial_i\partial^\alpha\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_i\partial^\alpha\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_i\partial^\alpha\widetilde{\mathsf{w}} + \overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w})\partial^\alpha\widetilde{\mathsf{w}} = \mathsf{h}^\alpha$$

$$\mathsf{h}^\alpha = \overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{f}\big) + \overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big) - \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{A}}'_i\big]\partial_i\widetilde{\mathsf{w}} - \overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\,\big]\widetilde{\mathsf{w}}$$

$$+ \sum_{i,j\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}^{\mathrm{d}}_{ij}\big]\partial_i\partial_j\widetilde{\mathsf{w}} + \sum_{i,j\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}^{\mathrm{c}}_{ij}\big]\partial_i\partial_j\widetilde{\mathsf{w}}.$$

  Multiply by $\partial^\alpha\delta\widetilde{\mathsf{w}}$, myltiply by $|\alpha|!/\alpha!$, integrate over $\mathbb{R}^d$, sum over $1 \leq |\alpha| \leq l'$, and add zeroth order estimate

Linearized Equations (8)

- **The l'th order inequality**
  - ⋆ Proceeding as for the zeroth order estimaet and use of $|\delta\widetilde{\mathsf{w}}|_{l'} \leq \mathsf{c}_1 E_{l'}(\delta\widetilde{\mathsf{w}})$

$$\partial_t E_{l'}^2(\delta\widetilde{\mathsf{w}}) + \delta_1 |\delta\widetilde{\mathsf{w}}_{\rm II}|_{l'+1}^2 \leq \mathsf{c}_2(1 + |\partial_t \mathsf{w}|_{l-2}) E_{l'}^2(\delta\widetilde{\mathsf{w}}) + \sum_{0\leq|\alpha|\leq l'} \frac{|\alpha|!}{\alpha!} \int_{\mathbb{R}^d} \langle \mathsf{h}^\alpha, \partial^\alpha \delta\widetilde{\mathsf{w}} \rangle \, d\boldsymbol{x}$$

  - ⋆ Right hand sides with $|\overline{\mathsf{A}}_0^{-1}\mathsf{f}|_{l'} \leq \mathsf{c}_1\big(1 + |\overline{\mathsf{A}}_0^{-1}(\mathsf{w}) - \overline{\mathsf{A}}_0^{-1}(\mathsf{w}^\star)|_l\big)\,|\mathsf{f}|_{l'} \leq \mathsf{c}_2|\mathsf{f}|_{l'}$ and eventual integration by parts for g

$$\Big|\int_{\mathbb{R}^d} \langle \overline{\mathsf{A}}_0 \partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{f}\big), \partial^\alpha \delta\widetilde{\mathsf{w}} \rangle \, d\boldsymbol{x}\Big| \leq |\overline{\mathsf{A}}_0|_\infty \, |\overline{\mathsf{A}}_0^{-1}\mathsf{f}|_{l'} \, |\delta\widetilde{\mathsf{w}}|_{l'} \leq \mathsf{c}_2|\mathsf{f}|_{l'} \, |\delta\widetilde{\mathsf{w}}|_{l'}$$

$$\Big|\int_{\mathbb{R}^d} \langle \overline{\mathsf{A}}_0 \partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big), \partial^\alpha \delta\widetilde{\mathsf{w}} \rangle \, d\boldsymbol{x}\Big| \leq \mathsf{c}_2|\mathsf{g}_{\rm II}|_{l'-1} \, |\delta\widetilde{\mathsf{w}}_{\rm II}|_{l'+1}$$

Linearized Equations (9)

- **The l'th order inequality**

  ⋆ Convective and dissipative contributions using commutator estimates

$$\Big|\int_{\mathbb{R}^d}\big\langle \overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{A}}_i'\big]\partial_i\widetilde{\mathsf{w}}, \partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\, d\boldsymbol{x}\Big| \le \mathsf{c}_2|\delta\widetilde{\mathsf{w}}|_{l'}^2$$

$$\Big|\int_{\mathbb{R}^d}\big\langle \overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}_{ij}^{\mathrm{d}}\big]\partial_i\partial_j\widetilde{\mathsf{w}}, \partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\, d\boldsymbol{x}\Big| \le \mathsf{c}_2\;|\delta\widetilde{\mathsf{w}}_{\textsc{ii}}|_{l'+1}\;|\delta\widetilde{\mathsf{w}}_{\textsc{ii}}|_{l'}$$

$$\sum_{0\le|\alpha|\le l'}\big|[\partial^\alpha, u]v\big|_0 \le \mathsf{c}_0|\nabla u|_{\bar{l}-1}|v|_{l'-1} \qquad \nabla u\in H^{\bar{l}-1}\quad v\in H^{l'-1}\quad \bar{l}\ge l_0+1$$

  ⋆ Block evaluation for the antisymmetric terms. The $(\textsc{i},\textsc{i})$ terms vanish and the $(\textsc{i},\textsc{ii})$ and $(\textsc{ii},\textsc{ii})$ are estimated with the commutator estimates

$$-\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^d}\big\langle \overline{\mathsf{A}}_0\big[\partial^\alpha, (\overline{\mathsf{A}}_0)^{-1}\,\overline{\mathsf{B}}_{ij}^{\mathrm{c}}\big]\partial_i\partial_j\delta\widetilde{\mathsf{w}}, \partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\, d\boldsymbol{x}$$

Linearized Equations (10)

- **The l'th order inequality**
  - ⋆ The (II,I) antisymmetric terms with $[\partial^\alpha, \mathfrak{V}]\partial_i\phi = \partial_i([\partial^\alpha, \mathfrak{V}]\phi) - [\partial^\alpha, \partial_i\mathfrak{V}]\phi$ are integration by parts

$$-\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^d}\big\langle \overline{\mathsf{A}}_0^{\text{II},\text{II}}\big[\partial^\alpha, (\overline{\mathsf{A}}_0^{\text{II},\text{II}})^{-1}\,\overline{\mathsf{B}}_{ij}^{\text{c}\,\text{II},\text{I}}\big]\partial_i\partial_j\delta\widetilde{\mathsf{w}}_{\text{I}}, \partial^\alpha\delta\widetilde{\mathsf{w}}_{\text{II}}\big\rangle\,d\boldsymbol{x} =$$

$$\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^d}\big\langle \big[\partial^\alpha, (\overline{\mathsf{A}}_0^{\text{II},\text{II}})^{-1}\,\overline{\mathsf{B}}_{ij}^{\text{c}\,\text{II},\text{I}}\big]\partial_j\delta\widetilde{\mathsf{w}}_{\text{I}}, \partial_i(\overline{\mathsf{A}}_0^{\text{II},\text{II}}\partial^\alpha\delta\widetilde{\mathsf{w}}_{\text{II}})\big\rangle\,d\boldsymbol{x}$$

$$+\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^d}\big\langle \overline{\mathsf{A}}_0^{\text{II},\text{II}}\big[\partial^\alpha, \partial_i\big((\overline{\mathsf{A}}_0^{\text{II},\text{II}})^{-1}\,\overline{\mathsf{B}}_{ij}^{\text{c}\,\text{II},\text{I}}\big)\big]\partial_j\delta\widetilde{\mathsf{w}}_{\text{I}}, \partial^\alpha\delta\widetilde{\mathsf{w}}_{\text{II}}\big\rangle\,d\boldsymbol{x}$$

Last sum estimated by using that $(\overline{\mathsf{A}}_0^{\text{II},\text{II}})^{-1}\,\overline{\mathsf{B}}_{ij}^{\text{c}\,\text{II},\text{I}}$ only depends on w_{r}

Upper bounds in the form $\mathsf{c}_2|\delta\widetilde{\mathsf{w}}|_{l'}\,|\delta\widetilde{\mathsf{w}}_{\text{II}}|_{l'+1} + \mathsf{c}_2|\nabla\mathsf{w}_{\text{r}}|_l|\delta\widetilde{\mathsf{w}}|_{l'}^2$

Linearized Equations (11)

- **The l'th order inequality**
  - ⋆ Terms associated with $\overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\,\big]\widetilde{\mathsf{w}}$ estimated as

$$\left|\int_{\mathbb{R}^d}\big\langle \overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\,\big]\widetilde{\mathsf{w}}, \partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\, d\boldsymbol{x}\right| \le \mathsf{c}_2|\nabla \mathsf{w}_{\mathrm{r}}|_l|\delta\widetilde{\mathsf{w}}|_{l'}^2$$

  since $\overline{\mathsf{L}} = \mathrm{diag}(\,\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}}\,)$ is a linear function of $\nabla\mathsf{w}_{\mathrm{r}}$

  - ⋆ Final differential inequality

$$\partial_t E_{l'}^2(\delta\widetilde{\mathsf{w}}) + \delta_1|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'+1}^2 \le \mathsf{c}_2\big(1 + |\partial_t\mathsf{w}|_{l-2} + |\nabla\mathsf{w}_{\mathrm{r}}|_l\big)E_{l'}^2(\delta\widetilde{\mathsf{w}}) + \mathsf{c}_2|\mathsf{f}|_{l'}E_{l'}(\delta\widetilde{\mathsf{w}}) + \mathsf{c}_2|\mathsf{g}_{\mathrm{II}}|_{l'-1}^2$$

  - ⋆ Apply Gronwall Lemma

Linearized Equations (12)

- **Regularized operators for $0 < \epsilon \leq 1$**

$$\mathsf{R}_\epsilon \phi(\mathbf{r}) = \int \mathfrak{a}_\epsilon(\mathbf{r} - \hat{\mathbf{r}})\phi(\hat{\mathbf{r}})\, d\hat{\mathbf{r}} \quad \mathfrak{a}_\epsilon = \epsilon^{-d}\mathfrak{a}(\mathbf{r}/\epsilon) \quad \int \mathfrak{a}\, d\mathbf{r} = 1 \quad \mathfrak{a} > 0 \text{ on } \mathrm{Ball}(0,1)$$

- **Regularized equations**

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i'(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}}$$
$$- \sum_{i,j\in\mathcal{D}} \mathsf{R}_\epsilon\overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\mathsf{w})\mathsf{R}_\epsilon\partial_i\partial_j\widetilde{\mathsf{w}} + \overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})\widetilde{\mathsf{w}} = \mathsf{f} + \mathsf{g}$$

- **Existence of solutions for linearized equations**

  Existence for regularized equations for ϵ fixed by uncoupling

  New estimates for solutions of regularized equations independent of ϵ

  Taking the limit $\epsilon \to 0$

Local existence Results for Diffuse Interface Models

- **Isothermal**

  Hattori and Li (1996) Danchin and Desjardins (2001) Kotschote (2008)
  Bresch et al. (2003) (2019)

- **Euler-Korteweg**

  Bresch et al. (2008) (2019) Benzoni et al. (2005) (2006) (2007)
  Donatelli et al. (2004) (2014) Tzavaras et al. (2018) (2017)

- **Full model**

  Haspot (2009) Kotschote (2012) (2014)

- **Symmetrization for diffuse interface fluids**

  Gavrilyuk and Gouin (2000) Kawashima et al. (2022)

Local Existence of Strong Solutions (1)

- **Structural assumptions**

Augmented system in normal form with the gradient constraint

Linearized equations enforcing the gradient constraint

$$\big(\overline{\mathsf{A}}'_i(\mathsf{w}) - \overline{\mathsf{A}}_i(\mathsf{w})\big)\boldsymbol{\nabla}\mathsf{w} + \overline{\mathsf{L}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}_{\mathrm{r}})\mathsf{w} + \mathsf{h}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}) = \mathsf{h}'(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$$

Right hand sides in the form

$$\mathsf{h}_{\mathrm{I}} = \sum_{i\in\mathcal{D}} \overline{\mathrm{M}}_i^{\mathrm{I}}(\mathsf{w})\partial_i\mathsf{w}_{\mathrm{r}} + \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}^{\mathrm{I},\mathrm{I}}(\mathsf{w})\partial_i\mathsf{w}_{\mathrm{r}}\partial_j\mathsf{w}_{\mathrm{r}}$$

$$\mathsf{h}_{\mathrm{II}} = \sum_{i\in\mathcal{D}} \overline{\mathrm{M}}_i^{\mathrm{II}}(\mathsf{w})\partial_i\mathsf{w} + \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II}}(\mathsf{w})\partial_i\mathsf{w}\partial_j\mathsf{w}$$

w_{r} is the more regular part $\mathsf{w}_{\mathrm{r}} = (\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{II}})^t$ of the normal vartiable

Local Existence of Strong Solutions (2)

Theorem 4. *Let* $d \geq 1$, $l \geq l_0 + 2$, $l_0 = [d/2] + 1$, *and let* $b > 0$.
Let $\mathcal{O}_0 \subset \overline{\mathcal{O}}_0 \subset \mathcal{O}_\mathsf{w}$, $0 < a_1 < \mathrm{dist}(\overline{\mathcal{O}}_0, \partial\mathcal{O}_\mathsf{w})$, $\mathcal{O}_1 = \{ \mathsf{w} \in \mathcal{O}_\mathsf{w}; \mathrm{dist}(\mathsf{w}, \overline{\mathcal{O}}_0) < a_1 \}$.
There exists $\bar{\tau}(\mathcal{O}_1, b) > 0$ *such that for any* w_0 *with* $\mathsf{w}_0 \in \mathcal{O}_0$, $\mathsf{w}_0 - \mathsf{w}^\star \in H^l$, $\mathsf{w}_{0\mathrm{I}''} = \nabla \mathsf{w}_{0\mathrm{I}'}$ *and*

$$|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 < b^2,$$

there exists a unique local solution w *with initial condition* $\mathsf{w}(0, \boldsymbol{x}) = \mathsf{w}_0(\boldsymbol{x})$, *such that* $\mathsf{w}(t, \boldsymbol{x}) \in \mathcal{O}_1$ *for* $(t, \boldsymbol{x}) \in [0, \bar{\tau}] \times \mathbb{R}^d$, $\mathsf{w}_{\mathrm{I}''} = \nabla \mathsf{w}_{\mathrm{I}'}$, *and*

$$\mathsf{w}_\mathrm{I} - \mathsf{w}_\mathrm{I}^\star \in C^0\big([0, \bar{\tau}], H^l\big) \cap C^1\big([0, \bar{\tau}], H^{l-2}\big)$$

$$\mathsf{w}_\mathrm{II} - \mathsf{w}_\mathrm{II}^\star \in C^0\big([0, \bar{\tau}], H^l\big) \cap C^1\big([0, \bar{\tau}], H^{l-2}\big) \cap L^2\big((0, \bar{\tau}), H^{l+1}\big)$$

Moreover, there exists $\mathsf{c}_{\mathrm{loc}}(\mathcal{O}_1, b) \geq 1$ *such that*

$$\sup_{0 \leq \tau \leq \bar{\tau}} |\mathsf{w}(\tau) - \mathsf{w}^\star|_l^2 + \int_0^{\bar{\tau}} |\mathsf{w}_\mathrm{II}(\tau) - \mathsf{w}_\mathrm{II}^\star|_{l+1}^2 d\tau \leq \mathsf{c}_{\mathrm{loc}}^2 |\mathsf{w}_0 - \mathsf{w}^\star|_l^2.$$

Local Existence of Strong Solutions (3)

- **Sketch of the proof (1)**

  $\star$ $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1, \overline{M}\,\big)$ defined by $\mathsf{w} - \mathsf{w}^\star \in C^0\big([0,\bar\tau], H^l\big)$, $\partial_t \mathsf{w} \in C^0\big([0,\bar\tau], H^{l-2}\big)$, $\mathsf{w}_{\rm II} - \mathsf{w}^\star_{\rm II} \in L^2\big((0,\bar\tau), H^{l+1}\big)$, $\mathsf{w}(t,\boldsymbol{x}) \in \mathcal{O}_1$, $\mathsf{w}_{\rm I''} = \nabla \mathsf{w}_{\rm I'}$, and

$$\sup_{0\le\tau\le\bar\tau} |\mathsf{w}(\tau) - \mathsf{w}^\star|_l^2 + \int_0^{\bar\tau} |\mathsf{w}_{\rm II}(\tau) - \mathsf{w}^\star_{\rm II}|_{l+1}^2\, d\tau \le \overline{M}^{\,2}$$

$$\int_0^{\bar\tau} |\partial_t \mathsf{w}(\tau)|_{l-2}^2\, d\tau \le \overline{M}^{\,2} \qquad \int_0^{\bar\tau} |\nabla \mathsf{w}_{\rm r}(\tau)|_l^2\, d\tau \le \overline{M}^{\,2}$$

  $\star$ $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1, \overline{M}\,\big)$ invariant by the map $\mathsf{w} \mapsto \widetilde{\mathsf{w}}$ for suitable $\overline{M}$ and $\bar\tau$ small enough

  Rely on a priori estimates for linearized equations applied to $\widetilde{\mathsf{w}}^k$

  Successive approximations $\{\mathsf{w}^k\}_{k\ge 0}$ with $\mathsf{w}^0 = \mathsf{w}^\star$, $\mathsf{w}^{k+1} = \widetilde{\mathsf{w}}^k$ well defined

Local Existence of Strong Solutions (4)

- **Sketch of the proof (2)**

  ⋆ The sequence $\{\mathsf{w}^k\}_{k\geq 0}$ is convergent over $[0,\bar\tau]$ for the norm

  $$\sup_{0\leq\tau\leq\bar\tau} |\delta\widetilde{\mathsf{w}}(\tau)|_{l-2}^2 + \int_0^{\bar\tau} |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}(\tau)|_{l-1}^2\, d\tau$$

  Rely on a priori estimates for linearized equations applied to $\mathsf{w}^{k+1} - \mathsf{w}^{k+1}$

  ⋆ $\mathsf{w}^k \to \overline{\mathsf{w}} \in C^0\big([0,\bar\tau], H^{l-2}\big)$ that is a solution (fixed point)

  $\overline{\mathsf{w}} \in L^\infty\big((0,\bar\tau), H^l\big)$ and $\overline{\mathsf{w}}_{\mathrm{II}} - \mathsf{w}_{\mathrm{II}}^\star \in L^2\big((0,\bar\tau), H^{l+1}\big)$

  ⋆ $\overline{\mathsf{w}} \in C^0\big((0,\bar\tau), H^l\big)$ since the sequence

  $$\mathsf{w}^\delta = \mathsf{R}_\delta \overline{\mathsf{w}}$$

  form a Cauchy sequence in $C^0\big([0,\bar\tau], H^l\big)$

Local Existence of Strong Solutions (5)

- **Application to diffuse interface fluids**

Theorem 5. *Let* $d \geq 1$, $l \geq l_0 + 2$, *and* $b > 0$. *There exists* $\bar{\tau}(\mathcal{O}_1, b) > 0$ *such that for any* w_0 *with* $\mathsf{w}_0 \in \overline{\mathcal{O}}_0$, $\mathsf{w}_0 - \mathsf{w}^\star \in H^l$, $\boldsymbol{w}_0 = \boldsymbol{\nabla} \rho_0$ *and* $|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 < b^2$ *there exists a unique local solution* w *with* $\mathsf{w}(0, \boldsymbol{x}) = \mathsf{w}_0(\boldsymbol{x})$, $\mathsf{w}(t, \boldsymbol{x}) \in \mathcal{O}_1$, $\boldsymbol{w} = \boldsymbol{\nabla} \rho$, *and*

$$\rho - \rho^\star \in C^0\big([0, \bar{\tau}], H^{l+1}\big),$$

$$\boldsymbol{v} - \boldsymbol{v}^\star \in C^0\big([0, \bar{\tau}], H^l\big) \cap L^2\big((0, \bar{\tau}), H^{l+1}\big)$$

$$T - T^\star \in C^0\big([0, \bar{\tau}], H^l\big) \cap L^2\big((0, \bar{\tau}), H^{l+1}\big).$$

Moreover, there exists $\mathsf{c}_{\mathrm{loc}}(\mathcal{O}_1, b) \geq 1$ *such that*

$$\sup_{0 \leq \tau \leq \bar{\tau}} |\rho(\tau) - \rho^\star|_{l+1}^2 + \sup_{0 \leq \tau \leq \bar{\tau}} |\boldsymbol{v}(\tau) - \boldsymbol{v}^\star|_l^2 + \sup_{0 \leq \tau \leq \bar{\tau}} |T(\tau) - T^\star|_l^2 + \int_0^{\bar{\tau}} |\boldsymbol{v}(\tau) - \boldsymbol{v}^\star|_{l+1}^2 d\tau$$

$$+ \int_0^{\bar{\tau}} |T(\tau) - T^\star|_{l+1}^2 d\tau \leq \mathsf{c}_{\mathrm{loc}}^2 \Big(|\rho_0(\tau) - \rho^\star|_{l+1}^2 + |\boldsymbol{v}_0(\tau) - \boldsymbol{v}^\star|_l^2 + |T_0(\tau) - T^\star|_l^2 \Big)$$

5 Strict Dissipativity and Asymptotic Stability

Global existence Results for Diffuse Interface Models

- **Isothermal**

  Hattori and Li (1996) Bresch at al. (2003) Tsyganov (2008)
  Wang and Tan (2011) Haspot (2011) Chave and Haspot (2013)
  Tan and Zhang (2014) Chanat al. (2015) Bresch at al. (2019)
  Plaza and Valdovinos (2022) Kawashima et al. (2021)

- **Full model**

  Kotschote (2012,2014) Hattori and Li (2016) Hou, Peng and Zhou (2018)
  Kawashima et al. (2022)

- **Strict dissipativity**

  Humpherys (2000) Plaza and Valdovinos (2022) Kawashima et al. (2022)

Strict Dissipativity (1)

- **Stability around a stable equilibrium state $\mathbf{w}^\star$ (1d) (Humpherys)**

  Linearized equations around $\mathsf{w}^\star$ with constant coefficients

$$\overline{\mathsf{A}}_0^\star \partial_t \mathsf{w} + \sum_{0\leq k\leq n} \overline{\mathsf{B}}_k^\star \partial^k \mathsf{w} = 0$$

  Fourier transform and eigenvalue problem

$$\Big(\lambda \overline{\mathsf{A}}_0^\star + \sum_{0\leq k\leq n} (\mathrm{i}\xi)^k\, \overline{\mathsf{B}}_k^\star\Big)\widehat{\mathsf{w}} = 0$$

  Decomposition $\overline{\mathsf{A}}^\star = \sum_{k\,\mathrm{odd}} (\mathrm{i}\xi)^{k-1} \overline{\mathsf{B}}_k^\star \widehat{\mathsf{w}}$ $\quad$ $\overline{\mathsf{B}}^\star = \sum_{k\,\mathrm{even}} (-1)^{k/2}\, \xi^k \overline{\mathsf{B}}_k^\star \widehat{\mathsf{w}}$

$$\big(\lambda \overline{\mathsf{A}}_0^\star + \mathrm{i}\xi \overline{\mathsf{A}}^\star(\xi) + \overline{\mathsf{B}}^\star(\xi)\big)\widehat{\mathsf{w}} = 0$$

  $\overline{\mathsf{A}}_0^\star$ symmetric and positive definite

  $\overline{\mathsf{A}}^\star(\xi)$ symmetric of constant multiplicity in ξ

  $\overline{\mathsf{B}}^\star(\xi)$ symmetric and positive semi-definite

Strict Dissipativity (2)

- **Equivalent conditions of strict stability**

  The system is strictly dissipative $\Re\big(\lambda(\xi)\big) < 0$ if $\xi \neq 0$

  The system is genuinely coupled, that is, there are no eigenvector of the matrix $\overline{\mathsf{A}}^\star(\xi)$ in $N\big(\overline{\mathsf{B}}^\star(\xi)\big)$ if $\xi \neq 0$

  There exists a–real analytic—matrix $\mathsf{K}(\xi)$ with $\mathsf{K}(\xi)\overline{\mathsf{A}}_0^\star$ skew Hermitian and

$$[\mathsf{K}(\xi), \overline{\mathsf{A}}^\star(\xi)] + \overline{\mathsf{B}}^\star(\xi) > 0 \qquad \xi \neq 0$$

- **The case particular case of second order systems (Kawashima et al.)**

  Classical results of Kawashima-Shizuta are recovered

  One can usually find explicitly $\mathsf{K}(\xi)$ in the form $\mathsf{K}(\xi) = \sum_{j\in\mathcal{D}} \xi_j \mathsf{K}_j$

Strict Dissipativity (3)

- **The case of third order systems of Korteweg type (Kawashima et al.)**

Linearized equations around a constant equilibrium state $\mathsf{w}^\star$

$$\overline{\mathsf{A}}_0^\star \partial_t \mathsf{w} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i^\star \partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^\star \partial_i \partial_j \mathsf{w} - \sum_{i,j,k\in\mathcal{D}} \overline{\mathsf{C}}_{ijk}^\star \partial_i \partial_j \partial_k \mathsf{w} = 0$$

Fourier transform and eigenvalue problem

$$\big(\overline{\mathsf{A}}_0^\star + i|\xi|\overline{\mathsf{A}}^\star(\omega) + |\xi|^2\overline{\mathsf{B}}^\star(\omega) + \mathrm{i}|\xi|^3\overline{\mathsf{C}}^\star(\omega)\big)\widehat{\mathsf{w}} = 0$$

$\overline{\mathsf{A}}^\star(\omega) = \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i^\star \omega_i \quad \overline{\mathsf{B}}^\star(\omega) = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^\star \omega_i\omega_j$
$\overline{\mathsf{C}}^\star(\omega) = \sum_{i,j,k\in\mathcal{D}} \overline{\mathsf{C}}_{ijk}^\star \omega_i\omega_j\omega_k$

- **Decomposition between dissipative and cohesive—dispersive—effects**

$$\mathsf{w} = (\rho, \boldsymbol{v}, T)^t \qquad (\mathbb{I}-Q_0)\mathsf{w} = (\mathbb{I}-Q)\mathsf{w} = \rho \qquad (\mathbb{I}-P)\mathsf{w} = (\boldsymbol{v}, T)^t$$

Strict Dissipativity (4)

- **The case of third order systems of Korteweg type (Kawashima et al.)**

  (B) $\overline{\mathsf{A}}_0^\star$ symmetric positive definite $\quad \overline{\mathsf{A}}^\star(\omega)$ symmetric

  $\overline{\mathsf{B}}^\star(\omega)$ symmetric positive semi-definite $\quad N\big(\overline{\mathsf{B}}^\star(\omega)\big)$ is independent of ω

  (S) There exists $S(\omega)$ with $S(\omega)\overline{\mathsf{A}}_0^\star$ symmetric positive semi-definite and

  $N\big(S(\omega)\overline{\mathsf{A}}_0^\star\big)$ is invariant, Q_0 projector onto $N\big(S(\omega)\overline{\mathsf{A}}_0^\star\big)$

  $S(\omega)\overline{\mathsf{A}}^\star(\omega) + \overline{\mathsf{C}}^\star(\omega)$ and $S(\omega)\overline{\mathsf{C}}^\star(\omega)$ are symmetric

  $\{S(\omega)\overline{\mathsf{B}}^\star(\omega)\}^{\rm sy}$ symmetric positive semi-definite

  (K) There exists K such that $\mathsf{K}(\omega)\overline{\mathsf{A}}_0^\star$ is skew-symmetric and

  $\{\mathsf{K}(\omega)\overline{\mathsf{A}}^\star(\omega)\}^{\rm sy} + \overline{\mathsf{B}}(\omega)$ positive definite

  $\{\mathsf{K}(\omega)\overline{\mathsf{C}}^\star(\omega)\}^{\rm sy} + \overline{\mathsf{B}}^\star(\omega)$ positive semi-definite,

  $N\big(\{\mathsf{K}(\omega)\overline{\mathsf{C}}^\star(\omega)\}^{\rm sy} + \overline{\mathsf{B}}^\star(\omega)\big) \subset N\big(S(\omega)\overline{\mathsf{A}}_0^\star\big)$

  Q projector onto $N\big(\{\mathsf{K}(\omega)\overline{\mathsf{C}}^\star(\omega)\}^{\rm sy} + \overline{\mathsf{B}}^\star(\omega)\big)$ that is invariant

Strict Dissipativity (5)

- **Hyperbolic-parabolic-dispersive estimates (Kawashima et al.)**

  Energy estimates in Fourier space $\quad \Re\big(\lambda(\xi)\big) \le -\delta|\xi|^2/(1+|\xi|^2)$

$$|\widehat{\mathsf{w}}(t,\xi)|^2+|\xi|^2|(\mathbb{I}-Q_0)\widehat{\mathsf{w}}(t,\xi)|^2+\frac{|\xi|^2}{1+|\xi|^2}\int_0^t\big(|\widehat{\mathsf{w}}(\tau,\xi)|^2+|\xi|^2|(\mathbb{I}-Q)\widehat{\mathsf{w}}(\tau,\xi)|^2\big)\,d\tau$$

$$+\int_0^t|\xi|^2|(\mathbb{I}-P)\widehat{\mathsf{w}}(\tau,\xi)|^2\,d\tau\le C\Big(|\widehat{\mathsf{w}}_0(t,\xi)|^2+|\xi|^2|(\mathbb{I}-Q_0)\widehat{\mathsf{w}}_0(t,\xi)|^2\Big)$$

- **Application to diffuse interface fluids**

$$\mathsf{w}=(\rho,\boldsymbol{v},T)^t \qquad (\mathbb{I}-Q_0)\mathsf{w}=(\mathbb{I}-Q)\mathsf{w}=\rho \qquad (\mathbb{I}-P)\mathsf{w}=\mathsf{w}_{\text{II}}=(\boldsymbol{v},T)^t$$

$$|\mathsf{w}(t)-\mathsf{w}^\star|_l^2+|\nabla\rho(t)|_l^2+\int_0^t\big(|\nabla\rho|_{l-1}^2+|\Delta\rho|_{l-1}^2+|\nabla\mathsf{w}_{\text{II}}|_l^2\big)\,d\tau$$

$$\le\bar{c}^2\Big(|\mathsf{w}_0-\mathsf{w}^\star|_l^2+|\nabla\rho_0(t)|_l^2\Big),$$

Strict Dissipativity (6)

- **Stronger assumptions for third order systems (Kawashima et al.)**

(B) $\overline{\mathsf{A}}_0^\star$ symmetric positive definite $\quad \overline{\mathsf{A}}_i^\star[](\omega)$ symmetric

$\overline{\mathsf{B}}^\star(\omega)$ symmetric positive semi-definite $\quad N\big(\overline{\mathsf{B}}^\star(\omega)\big)$ is independent of ω

(S) There exists $S(\omega)$ with $S(\omega)\overline{\mathsf{A}}_0^\star$ symmetric positive semi-definite

$N\big(S(\omega)\overline{\mathsf{A}}_0^\star\big)$ is invariant, Q_0 projector onto $N\big(S(\omega)\overline{\mathsf{A}}_0^\star\big)$

$S(\omega)\overline{\mathsf{A}}^\star(\omega) + \overline{\mathsf{C}}(\omega)$ and $S(\omega)\overline{\mathsf{C}}^\star(\omega)$ are symmetric

$S(\omega)\overline{\mathsf{B}}^\star(\omega)$ symmetric positive semi-definite

(K') There exists K such that $\mathsf{K}(\omega)\overline{\mathsf{A}}_0^\star$ is skew-symmetric

$\{\mathsf{K}(\omega)\overline{\mathsf{A}}^\star((\omega)\}^{\mathrm{sy}} + \overline{\mathsf{B}}^\star(\omega)$ positive definite

$\{\mathsf{K}(\omega)\overline{\mathsf{C}}^\star(\omega)\}^{\mathrm{sy}}$ positive semi-definite, $N\big(\{\mathsf{K}(\omega)\overline{\mathsf{C}}^\star(\omega)\}^{\mathrm{sy}}\big) \subset N\big(S(\omega)\overline{\mathsf{A}}_0^\star\big)$

$N\big(\{\mathsf{K}(\omega)\overline{\mathsf{C}}^\star(\omega)\}^{\mathrm{sy}}\big)$ is invariant, Q' projector onto $N\big(\{\mathsf{K}(\omega)\overline{\mathsf{C}}^\star(\omega)\}^{\mathrm{sy}}\big)$

(A) $Q_0\mathsf{K}(\omega)\overline{\mathsf{A}}_0^\star Q_0 = 0 \quad Q'\mathsf{K}(\omega)\overline{\mathsf{B}}_{ij}^\star(\omega) = 0$

Strict Dissipativity (7)

- **Parabolic-Dispersive type estimates (Kawashima et al.)**

  Energy estimates in Fourier space $\Re\big(\lambda(\xi)\big) \leq -\delta|\xi|^2$

$$|\widehat{\mathsf{w}}(t,\xi)|^2+|\xi|^2|(\mathbb{I}-Q_0)\widehat{\mathsf{w}}(t,\xi)|^2+|\xi|^2\int_0^t \big(|\widehat{\mathsf{w}}(\tau,\xi)|^2+|\xi|^2|(\mathbb{I}-Q')\widehat{\mathsf{w}}(\tau,\xi)|^2\big)\,d\tau$$

$$+\int_0^t |\xi|^2|(\mathbb{I}-P)\widehat{\mathsf{w}}(\tau,\xi)|^2\,d\tau \leq C\Big(|\widehat{\mathsf{w}}_0(t,\xi)|^2+|\xi|^2|(\mathbb{I}-Q_0)\widehat{\mathsf{w}}_0(t,\xi)|^2\Big)$$

- **Application to diffuse interface fluids**

$$\mathsf{w}=(\rho,\boldsymbol{v},T)^t \qquad (\mathbb{I}-Q_0)\mathsf{w}=(\mathbb{I}-Q')\mathsf{w}=\rho \qquad (\mathbb{I}-P)\mathsf{w}=(\boldsymbol{v},T)^t$$

$$|\mathsf{w}(t)-\mathsf{w}^\star|_l^2+|\nabla\rho(t)|_l^2+\int_0^t\big(|\nabla\rho|_l^2+|\Delta\rho|_l^2+|\nabla\mathsf{w}_{\text{II}}|_l^2\big)\,d\tau$$

$$\leq \bar{c}^2\Big(|\mathsf{w}_0-\mathsf{w}^\star|_l^2+|\nabla\rho_0(t)|_l^2\Big),$$

Strict Dissipativity (8)

- **Simplified strict dissipativity for anugmented systems**

  There exists K_j, $j \in \mathcal{D}$, with $\big(\mathsf{K}_j \overline{\mathsf{A}}_0^\star\big)^t = -\mathsf{K}_j \overline{\mathsf{A}}_0^\star \quad \mathsf{K}(\omega) = \sum_{j\in\mathcal{D}} \xi_j \mathsf{K}_j$

  such that for any ϕ regular with $\phi_{\mathrm{I}''} = \nabla \phi_{\mathrm{I}'}$

$$\sum_{i,j\in\mathcal{D}} \int \langle \partial_j \phi,\, \mathsf{K}_j \overline{\mathsf{A}}_i^\star \partial_i \phi \rangle d\boldsymbol{x} \;-\; \sum_{i,j,j'\in\mathcal{D}} \int \langle \partial_j \phi,\, \mathsf{K}_j \overline{\mathsf{B}}_{ij'}^{\mathsf{c}\star} \partial_i \partial_{j'} \phi \rangle d\boldsymbol{x}$$

$$\geq \delta \Big(\int |\nabla \phi_{\mathrm{I}'}|^2 d\boldsymbol{x} + \int |\Delta \phi_{\mathrm{I}'}|^2 d\boldsymbol{x} \Big) - \mathsf{c} \int |\nabla \phi_{\mathrm{II}}|^2 d\boldsymbol{x}.$$

- **Decomposition between convection and cohesive terms**

$$\sum_{i,j\in\mathcal{D}} \int \langle \partial_j \phi,\, \mathsf{K}_j \overline{\mathsf{A}}_i^\star \partial_i \phi \rangle d\boldsymbol{x} \geq \delta \int |\nabla \phi_{\mathrm{I}'}|^2 d\boldsymbol{x} - \mathsf{c} \int |\nabla \phi_{\mathrm{II}}|^2 d\boldsymbol{x}$$

$$- \sum_{i,j,j'\in\mathcal{D}} \int \langle \partial_j \phi,\, \mathsf{K}_j \overline{\mathsf{B}}_{ij'}^{\mathsf{c}\star} \partial_i \partial_{j'} \phi \rangle d\boldsymbol{x} \geq \delta \int |\Delta \phi_{\mathrm{I}'}|^2 d\boldsymbol{x}$$

Strict Dissipativity (9)

- **Algebraic condition for convective matrices**

  $\sum_{i,j\in\mathcal{D}} \xi_i\xi_j \mathsf{K}_j \overline{\mathsf{A}}_i^\star + \sum_{i,j\in\mathcal{D}} \xi_i\xi_j \overline{\mathsf{B}}_{ij}^\star$ is positive definite.

- **Algebraic condition for cohesive matrices**

$$-\sum_{i,j,j'\in\mathcal{D}} \int \langle \partial_j\phi, \, \mathsf{K}_j \overline{\mathsf{B}}^{c\star}_{ij'} \partial_i\partial_{j'}\phi \rangle d\boldsymbol{x} = -\sum_{i,j,j',l\in\mathcal{D}} \int \langle \partial_j \phi_{\mathrm{I}'}, \, (\mathsf{K}_j \overline{\mathsf{B}}^{c\star}_{ij'})_{1,1+l} \partial_i\partial_{j'}\partial_l \phi_{\mathrm{I}'} \rangle d\boldsymbol{x}$$

$$= \sum_{i,j,j',l\in\mathcal{D}} \int \langle \partial_i\partial_j \phi_{\mathrm{I}'}, \, (\mathsf{K}_j \overline{\mathsf{B}}^{c\star}_{ij'})_{1,1+l} \partial_{j'}\partial_l \phi_{\mathrm{I}'} \rangle d\boldsymbol{x}$$

$$\sum_{i,j,j',l\in\mathcal{D}} \xi_j\xi_l (\mathsf{K}_j \overline{\mathsf{B}}^{c\star}_{ij'} \partial_i\partial_{j'}\phi)_{1,1+l} \bar{\xi}_i \bar{\xi}_{j'} \geq \delta |\xi|^4 \qquad (\mathsf{K}_j \overline{\mathsf{B}}^{c\star}_{ij'} \partial_i\partial_{j'}\phi)_{1,1+l} = \delta_{ij}\delta_{j'l}\delta$$

Global Existence and Asymptotic Stability (1)

- **Normal form around a stable state**

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t \mathsf{w} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w})\partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j \mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j \mathsf{w} = \mathsf{h}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$$

- **Properties of the normal form**

$\overline{\mathsf{A}}_0 = \mathrm{diag}(\overline{\mathsf{A}}_0^{\mathrm{I,I}}, \overline{\mathsf{A}}_0^{\mathrm{II,II}})$ symmetric positive definite $\qquad \overline{\mathsf{A}}_i$ symmetic for $i \in \mathcal{D}$

$(\overline{\mathsf{B}}^{\mathrm{d}}_{ij})^t = \overline{\mathsf{B}}^{\mathrm{d}}_{ji} \quad \overline{\mathsf{B}}^{\mathrm{d}}_{ij} = \mathrm{diag}(0, \overline{\mathsf{B}}^{\mathrm{d\,II,II}}_{ij}) \quad \overline{\mathsf{B}}^{\mathrm{d\,II,II}} = \sum_{i,j\in\mathcal{D}} \xi_i\xi_j \overline{\mathsf{B}}^{\mathrm{d\,II,II}}_{ij}$ positive definite

$(\overline{\mathsf{B}}^{\mathrm{c}}_{ij})^t = -\overline{\mathsf{B}}^{\mathrm{c}}_{ji} \quad \overline{\mathsf{B}}^{\mathrm{c\,I,I}}_{ij} = 0 \quad \overline{\mathsf{B}}^{\mathrm{c\,I,II}}_{ij},\ \overline{\mathsf{B}}^{\mathrm{c\,II,I}}_{ij},\ \overline{\mathsf{A}}_0^{\mathrm{II,II}}$ depend on $\mathsf{w}_{\mathrm{r}} = (\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{II}})^t$

$\mathsf{h} = (\mathsf{h}_{\mathrm{I}}, \mathsf{h}_{\mathrm{II}})^t \qquad \mathsf{h}_{\mathrm{I}} = \left(0, -\frac{\varkappa}{T}\sum_{i\in\mathcal{D}} w_i \boldsymbol{\nabla} v_i\right)^t \qquad \mathsf{h}_{\mathrm{II}} = \mathsf{h}_{\mathrm{II}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$

- **Strict stability of $\mathbf{w}^\star$**

Global Existence and Asymptotic Stability (2)

- **Global existence for augmented systems**

  Let $d \geq 1$ and $l \geq [d/2]+2$ be integers. There exists $\bar{b} > 0$ such that if $\mathsf{w}_0 - \mathsf{w}^\star \in H^l(\mathbb{R}^d)$ and

  $$|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 < \bar{b}^2,$$

  there exists a unique global solution to the Cauchy problem

  $$\overline{\mathsf{A}}_0 \partial_t \mathsf{w} + \sum_{i \in \mathcal{D}} \overline{\mathsf{A}}_i \partial_i \mathsf{w} - \sum_{i,j \in \mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{d}} \partial_i \partial_j \mathsf{w} - \sum_{i,j \in \mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{c}} \partial_i \partial_j \mathsf{w} + \overline{\mathsf{L}} \mathsf{w} = \mathsf{h}(\mathsf{w}, \boldsymbol{\nabla} \mathsf{w})$$

  with initial condition $\mathsf{w}(0, \boldsymbol{x}) = \mathsf{w}_0(\boldsymbol{x})$ and regularity

  $$\mathsf{w}_{\mathrm{I}} - \mathsf{w}_{\mathrm{I}}^\star \in C^0\big([0,\infty), H^l\big) \cap C^1\big([0,\infty), H^{l-1}\big), \qquad \partial_{\boldsymbol{x}} \mathsf{w}_{\mathrm{I}} \in L^2\big((0,\infty), H^{l-1}\big),$$

  $$\mathsf{w}_{\mathrm{II}} - \mathsf{w}_{\mathrm{II}}^\star \in C^0\big([0,\infty), H^l\big) \cap C^1\big([0,\infty), H^{l-2}\big), \qquad \partial_{\boldsymbol{x}} \mathsf{w}_{\mathrm{II}} \in L^2\big((0,\infty), H^l\big).$$

Global Existence and Asymptotic Stability (3)

- **Asymptotic stability of hyperbolic-parabolic type**

  There exists also a constant $\bar{c}$ such that w satisfies the estimate

$$|\mathsf{w}(t) - \mathsf{w}^\star|_l^2 + \int_0^t \left(|\nabla \mathsf{w}_{\mathrm{I}}|_{l-1}^2 + |\nabla \mathsf{w}_{\mathrm{II}}|_l^2 \right) d\tau \leq \bar{c}^2 |\mathsf{w}_0 - \mathsf{w}^\star|_l^2$$

  and $\sup_{\boldsymbol{x} \in \mathbb{R}^d} |\mathsf{w}(t, \boldsymbol{x}) - \mathsf{w}^\star|$ goes to zero as $t \to \infty$.

- **Stronger results for Korteweg**

  There exists also a constant $\bar{c}$ such that w satisfies the estimate

$$|\mathsf{w}(t) - \mathsf{w}^\star|_l^2 + \int_0^t (|\nabla \mathsf{w}_{\mathrm{I}}|_l^2 + |\nabla \mathsf{w}_{\mathrm{II}}|_l^2) \, d\tau \leq \bar{c}^2 |\mathsf{w}_0 - \mathsf{w}^\star|_l^2$$

6 Fluid Mixtures of Korteweg type

Fluid Mixtures of Korteweg types (1)

- **Cahn-Hilliard fluid mixture with equal mass capillarities**

$$\partial_t \rho_i + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho_i \boldsymbol{v}) + \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mathcal{F}}_i = m_i \omega_i \qquad i \in \mathfrak{S} = \{1, \dots, \mathsf{n}_\mathrm{s}\}$$

$$\partial_t(\rho \boldsymbol{v}) + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \boldsymbol{v} \otimes \boldsymbol{v}) + \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mathcal{P}} = 0$$

$$\partial_t\big(\mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2\big) + \boldsymbol{\nabla}\boldsymbol{\cdot}\big(\boldsymbol{v}(\mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2)\big) + \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\mathcal{Q}} + \boldsymbol{\mathcal{P}}\boldsymbol{\cdot}\boldsymbol{v}) = 0$$

- **Structure of diffusion fluxes, pressure tensor and heat flux**

$$\boldsymbol{\mathcal{F}}_i = \boldsymbol{\mathcal{F}}_i^\mathrm{d} \qquad i \in \mathfrak{S}$$

$$\boldsymbol{\mathcal{P}} = p\boldsymbol{I} + \varkappa \boldsymbol{\nabla}\rho \otimes \boldsymbol{\nabla}\rho - \rho \boldsymbol{\nabla}\boldsymbol{\cdot}(\varkappa \boldsymbol{\nabla}\rho)\boldsymbol{I} + \boldsymbol{\mathcal{P}}^\mathrm{d}$$

$$\boldsymbol{\mathcal{Q}} = \varkappa \rho \boldsymbol{\nabla}\rho \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} + \boldsymbol{\mathcal{Q}}^\mathrm{d}$$

- **Extended thermodynamics**

$$p = p^\mathrm{cl}(\rho, T) - \tfrac{1}{2}\varkappa|\boldsymbol{\nabla}\rho|^2 \qquad \mathcal{E} = \mathcal{E}^\mathrm{cl}(\rho, T) + \tfrac{1}{2}(\varkappa - T\partial_T \varkappa)|\boldsymbol{\nabla}\rho|^2$$

$$g_i = g_i^\mathrm{cl} \qquad \text{Gibbs relation} \quad T\, d\mathcal{S} = d\mathcal{E} - \textstyle\sum_{i\in\mathfrak{S}} g_i\, d\rho_i - \varkappa \boldsymbol{\nabla}\rho \boldsymbol{\cdot} d\boldsymbol{\nabla}\rho$$

Fluid Mixtures of Korteweg types (2)

- **Thermodynamic form for multicomponent fluxes**

$$\boldsymbol{\mathcal{P}}^{\mathrm{d}} = -\,\mathfrak{v}\boldsymbol{\nabla\cdot v}\,\boldsymbol{I} - \eta\big(\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^{t} - \tfrac{2}{3}\boldsymbol{\nabla\cdot v}\,\boldsymbol{I}\big),$$

$$\boldsymbol{\mathcal{F}}_i = -\sum_{j\in\mathfrak{S}} L_{ij}\boldsymbol{\nabla}\Big(\frac{g_j}{T}\Big) - L_{i\mathrm{e}}\boldsymbol{\nabla}\Big(\frac{-1}{T}\Big),$$

$$\boldsymbol{\mathcal{Q}}^{\mathrm{d}} = -\sum_{i\in\mathfrak{S}} L_{\mathrm{e}i}\boldsymbol{\nabla}\Big(\frac{g_j}{T}\Big) - L_{\mathrm{ee}}\boldsymbol{\nabla}\Big(\frac{-1}{T}\Big),$$

- **Structure of the matrix L**

  $L = (L_{ij})_{i,j\in\mathfrak{S}\cup\{\mathrm{e}\}}$ symmetric positive semi-definite

  $N(L) = \mathrm{Span}(1,\ldots,1,0)^t$

- **Chemisty source terms**

  Classical framework of reactive multicomponent fluids

Fluid Mixtures of Korteweg types (3)

- **Thermodynamic stability**

  Assume $\mathsf{z}^{\mathrm{cl}} = (\rho_1, \dots, \rho_{\mathsf{n_s}}, T)^t \;\mapsto\; \mathsf{u}^{\mathrm{cl}} = (\rho_1, \dots, \rho_{\mathsf{n_s}}, \mathcal{E}^{\mathrm{cl}})^t$ locally invertible

  $$\partial^2_{\mathsf{u}^{\mathrm{cl}}\mathsf{u}^{\mathrm{cl}}} \mathcal{S}^{\mathrm{cl}} \;\text{ negative definite } \iff \partial_T \mathcal{E}^{\mathrm{cl}} > 0 \text{ and } \Lambda \text{ positive definite}$$

  $\Lambda = (\Lambda)_{i,j\in\mathfrak{S}} \qquad \Lambda_{ij} = \partial_{\rho_j} g_i / T$

- **Assumptions on thermodynamics**

($\mathsf{H}_1^{\mathsf{cl}}$) *$\mathcal{E}^{\mathrm{cl}}$, p^{cl}, and $\mathcal{S}^{\mathrm{cl}}$ are C^γ functions of $\mathsf{z}^{\mathrm{cl}} = (\rho_1, \dots, \rho_{\mathsf{n}_s},, T)^t$ over $\mathcal{O}_{\mathsf{z}^{\mathrm{cl}}}$ $\mathcal{O}_{\mathsf{z}^{\mathrm{cl}}} \subset (0,\infty)^{nspecies+1}$ simply connected nonempty open set. The map $(\rho_1, \dots, \rho_{\mathsf{n}_s}, T) \mapsto (\rho, \frac{g_2 - g1}{T}, \dots, \frac{g_{\mathsf{n}_s} - g1}{T}, T)^t$ is globally invertible*

($\mathsf{H}_2^{\mathsf{cl}}$) *Letting $\mathcal{G}^{\mathrm{cl}} = \mathcal{E}^{\mathrm{cl}} + p^{\mathrm{cl}} - T\mathcal{S}^{\mathrm{cl}} = \sum_{i\in\mathfrak{S}} \rho_i g_i^{\mathrm{cl}}$ then $T\,d\mathcal{S}^{\mathrm{cl}} = \,d\mathcal{E}^{\mathrm{cl}} - \sum_{i\in\mathfrak{S}} g_i^{\mathrm{cl}}\,d\rho_i$*

($\mathsf{H}_3^{\mathsf{cl}}$) *$\mathcal{O}_{\mathsf{z}^{\mathrm{cl}}}$ is increasing with T and $\partial_T \mathcal{E}^{\mathrm{cl}} > 0$*

Fluid Mixtures of Korteweg types (4)

- **Extra unknown** $\boldsymbol{w} = \boldsymbol{\nabla}\rho$

$$\partial_t \boldsymbol{w} + \sum_{i\in\mathcal{D}} \partial_i \big(\boldsymbol{w}\, v_i + \rho \boldsymbol{\nabla} v_i\big) = 0 \qquad \mathcal{D} = \{\, 1, \ldots, d \,\}$$

- **Augmented unknowns**

$$\mathsf{u} = \big(\rho_1, \ldots, \rho_{\mathsf{n_s}}, \boldsymbol{w}, \rho\boldsymbol{v}, \mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2\big)^t \qquad \mathsf{z} = \big(\rho_1, \ldots, \rho_{\mathsf{n_s}}, \boldsymbol{w}, \boldsymbol{v}, T\big)^t$$

- **New thermodynamic functions**

$$\mathcal{E} = \mathcal{E}^{\mathrm{cl}} + \tfrac{1}{2}(\varkappa - T\partial_T\varkappa)|\boldsymbol{w}|^2 \qquad \mathcal{S} = \mathcal{S}^{\mathrm{cl}} - \tfrac{1}{2}\partial_T\varkappa|\boldsymbol{w}|^2$$

$$p = p^{\mathrm{cl}} - \tfrac{1}{2}\varkappa|\boldsymbol{w}|^2 \qquad g = g^{\mathrm{cl}}$$

- **New convectives fluxes using the Legendre tranform of entropy**

Fluid Mixtures of Korteweg types (5)

- **Thermodynamic functions**

(H$_1$) $\mathcal{E}$, p, $\mathcal{S}$ *are* C^γ *functions of* $\mathsf{z} \in \mathcal{O}_\mathsf{z} \subset (0,\infty)^{\mathsf{n}_s} \times \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty)$ *open set and* $\varkappa = \varkappa(T)$ *is a* $C^{\gamma+1}$ *function of temperature* T *over* $\mathcal{O}_\mathsf{z}$

If $(\rho_1,\ldots,\rho_{\mathsf{n}_s},T)^t \in \mathcal{O}_{\mathsf{z}^{\mathrm{cl}}}$, $(\rho_1,\ldots,\rho_{\mathsf{n}_s},0,0,T)^t \in \mathcal{O}_\mathsf{z}$ *and If* $(\rho_1,\ldots,\rho_{\mathsf{n}_s},\boldsymbol{w},\boldsymbol{v},T)^t \in \mathcal{O}_\mathsf{z}$, $(\rho_1,\ldots,\rho_{\mathsf{n}_s},T)^t \in \mathcal{O}_{\mathsf{z}^{\mathrm{cl}}}$

(H$_2$) $\mathcal{G} = \mathcal{E} + p - T\mathcal{S} = \sum_{i\in\mathfrak{S}} \rho_i g_i \quad T\,d\mathcal{S} = d\mathcal{E} - \sum_{i\in\mathfrak{S}} g_i\,d\rho_i - \varkappa\boldsymbol{w}\cdot d\boldsymbol{w}$

(H$_3$) *The open set* $\mathcal{O}_\mathsf{z}$ *is increasing with temperature* T *and* $\partial_T\mathcal{E} > 0$

(H$_4$) *The capillarity coefficient is positive* $\varkappa > 0$ *over* $\mathcal{O}_\mathsf{z}$

(H$_5$) *The coefficients* $\mathfrak{v}$, η, *and the matrix* L *are* C^γ *functions over* $\mathcal{O}_\mathsf{z}$

We have $\eta > 0$, $\mathfrak{v} \geq 0$, $\mathfrak{v} + \eta(1-\frac{2}{d}) > 0$, L *is symmetric positive semi-definite and* $N(L) = \mathbb{R}(1,\ldots,1,0,0,0,0)^t$.

Fluid Mixtures of Korteweg types (6)

- **Normal variable**

$$\mathsf{w} = \Big(\rho, \boldsymbol{w}, \frac{g_2 - g_1}{T}, \ldots, \frac{g_{\mathsf{n_s}} - g_1}{T}, \boldsymbol{v}, T\Big)^t \qquad \mathsf{w} = (\mathsf{w}_{\mathrm{I}}, \mathsf{w}_{\mathrm{II}})^t$$

$$\mathsf{w}_{\mathrm{I}} = (\rho, \boldsymbol{w})^t \quad \mathsf{w}_{\mathrm{II}} = \Big(\frac{g_2 - g_1}{T}, \ldots, \frac{g_{\mathsf{n_s}} - g_1}{T}, \boldsymbol{v}, T\Big)^t$$

$\mathbb{R}^{\mathsf{n}} = \mathbb{R}^{\mathsf{n}_{\mathrm{I}}} \times \mathbb{R}^{\mathsf{n}_{\mathrm{II}}} \qquad \mathsf{n} = \mathsf{n}_{\mathrm{I}} + \mathsf{n}_{\mathrm{II}} \qquad \mathsf{n}_{\mathrm{I}} = d + 1 \qquad \mathsf{n}_{\mathrm{II}} = \mathsf{n_s} + d$

$\mathsf{z} \to \mathsf{w}$ diffeomorphism from $\mathcal{O}_{\mathsf{z}}$ onto $\mathcal{O}_{\mathsf{w}}$ and $\mathsf{u} \to \mathsf{w}$ from $\mathcal{O}_{\mathsf{u}}$ onto $\mathcal{O}_{\mathsf{w}}$

- **Normal form**

$\mathsf{u} = \mathsf{u}(\mathsf{w})$ and multiplication on the left by $(\partial_{\mathsf{w}}\mathsf{v})^t$

Stabilisation not required near a stable equilibrium state.

$\mathsf{w}_{\mathrm{I}} = (\mathsf{w}_{\mathrm{I'}}, \mathsf{w}_{\mathrm{I''}})^t \qquad \mathsf{w}_{\mathrm{I'}} = \rho \qquad \mathsf{w}_{\mathrm{I''}} = \boldsymbol{w} \qquad \boldsymbol{\nabla}\mathsf{w}_{\mathrm{I'}} = \mathsf{w}_{\mathrm{I''}} \qquad \mathsf{w}_{\mathrm{r}} = (\mathsf{w}_{\mathrm{I'}}, \mathsf{w}_{\mathrm{II}})^t$

Fluid Mixtures of Korteweg types (7)

- **Normal form**

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\mathsf{w} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w})\partial_i\mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j\mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j\mathsf{w} + \overline{\mathsf{L}}\mathsf{w} = \mathsf{h}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$$

- **Properties of the normal form**

$\overline{\mathsf{A}}_0 = \mathrm{diag}(\overline{\mathsf{A}}_0^{\mathrm{I,I}}, \overline{\mathsf{A}}_0^{\mathrm{II,II}})$ symmetric positive definite $\qquad \overline{\mathsf{A}}_i$ symmetic for $i \in \mathcal{D}$

$(\overline{\mathsf{B}}^{\mathrm{d}}_{ij})^t = \overline{\mathsf{B}}^{\mathrm{d}}_{ji} \qquad \overline{\mathsf{B}}^{\mathrm{d}}_{ij} = \mathrm{diag}(0, \overline{\mathsf{B}}^{\mathrm{d\,II,II}}_{ij}) \qquad \overline{\mathsf{B}}^{\mathrm{d\,II,II}} = \sum_{i,j\in\mathcal{D}} \xi_i\xi_j \overline{\mathsf{B}}^{\mathrm{d\,II,II}}_{ij}$ positive definite

$(\overline{\mathsf{B}}^{\mathrm{c}}_{ij})^t = -\overline{\mathsf{B}}^{\mathrm{c}}_{ji} \qquad \overline{\mathsf{B}}^{\mathrm{c\,I,I}}_{ij} = 0 \qquad \overline{\mathsf{B}}^{\mathrm{c\,I,II}}_{ij},\ \overline{\mathsf{B}}^{\mathrm{c\,II,I}}_{ij},\ \overline{\mathsf{A}}_0^{\mathrm{II,II}}$ depend on $\mathsf{w}_{\mathrm{r}} = (\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{II}})^t$

- **Right hand side**

$$\mathsf{h} = (\mathsf{h}_{\mathrm{I}}, \mathsf{h}_{\mathrm{II}})^t \qquad \mathsf{h}_{\mathrm{I}} = \Big(0, -\frac{\varkappa}{T}\sum_{i\in\mathcal{D}} w_i \boldsymbol{\nabla} v_i\Big)^t \qquad \mathsf{h}_{\mathrm{II}} = \mathsf{h}_{\mathrm{II}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$$

Fluid Mixtures of Korteweg types (8)

- **Results for mixtures of fluids of Korteweg type**

  Normal form with Strict dissipativity

  Gradient constraint satisfied as well as for proper linearized equations

  Global existence and asymptotic stability of stable equilibrium states such that $\partial_T \mathcal{E} > 0$ and $\det \Lambda > 0$

Conclusion/Future work

- **Physical aspects**

  Mixtures with polyatomic species with chemical reactions

  Numerical simulations at the Molecular/Boltzmann/Fluid levels

  Boundary equations at solid walls

- **Mathematical and numerical aspects aspects**

  Numerical simulations of subcritical to supercritical mixtures of fluids

  Stronger estimates for ρ and $\nabla\rho$ around equilibrium states

  Global existence results around stationary nonconstant equilibrium states

  Multicomponent mixtures and Cahn-Hilliard equations

Cahn–Hilliard Fluid Mixtures (1)

- **Cahn-Hilliard fluid mixtures form the kinetic theory**

$$\partial_t \rho_i + \boldsymbol{\nabla}\cdot(\rho_i \boldsymbol{v}) + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_i = 0, \qquad i \in \mathfrak{S},$$

$$\partial_t(\rho \boldsymbol{v}) + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}\otimes\boldsymbol{v}) + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} = 0,$$

$$\partial_t\big(\mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2\big) + \boldsymbol{\nabla}\cdot\big(\boldsymbol{v}(\mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2)\big) + \boldsymbol{\nabla}\cdot(\boldsymbol{\mathcal{Q}} + \boldsymbol{\mathcal{P}}\cdot\boldsymbol{v}) = 0,$$

- **Pressure tensor and heat flux**

$$\boldsymbol{\mathcal{P}} = p\boldsymbol{I} + \sum_{i,j\in\mathfrak{S}} \varkappa_{ij}\boldsymbol{\nabla}\rho_i\otimes\boldsymbol{\nabla}\rho_j - \sum_{i,j\in\mathfrak{S}} \rho_i\boldsymbol{\nabla}\cdot(\varkappa_{ij}\boldsymbol{\nabla}\rho_j) + \boldsymbol{\mathcal{P}}^{\mathrm{d}}$$

$$\boldsymbol{\mathcal{Q}} = \sum_{i,j\in\mathfrak{S}} \varkappa_{ij}\rho_i\boldsymbol{\nabla}\rho_j\boldsymbol{\nabla}\cdot\boldsymbol{v} + \sum_{i,j\in\mathfrak{S}} \varkappa_{ij}\boldsymbol{\nabla}\rho_j\,\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_i - \sum_{i,j\in\mathfrak{S}} \boldsymbol{\nabla}\cdot(\varkappa_{ij}\boldsymbol{\nabla}\rho_j)\boldsymbol{\mathcal{F}}_i + \boldsymbol{\mathcal{Q}}^{\mathrm{d}}$$

Cahn–Hilliard Fluid Mixtures (2)

- **Thermodynamic form for multicomponent fluxes**

$$\boldsymbol{\mathcal{P}}^{\mathrm{d}} = -\,\mathfrak{v}\boldsymbol{\nabla\cdot v}\,\boldsymbol{I} - \eta\big(\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^t - \tfrac{2}{3}\boldsymbol{\nabla\cdot v}\,\boldsymbol{I}\big),$$

$$\boldsymbol{\mathcal{F}}_i = -\sum_{j\in\mathfrak{S}} L_{ij}\Big(\boldsymbol{\nabla}\Big(\frac{g_j}{T}\Big) - \frac{\boldsymbol{\nabla}\,\boldsymbol{\nabla\cdot}\overline{\boldsymbol{\gamma}}_i}{T}\Big) - L_{i\mathrm{e}}\boldsymbol{\nabla}\Big(\frac{-1}{T}\Big),$$

$$\boldsymbol{\mathcal{Q}}^{\mathrm{d}} = -\sum_{i\in\mathfrak{S}} L_{\mathrm{e}i}\Big(\boldsymbol{\nabla}\Big(\frac{g_j}{T}\Big) - \frac{\boldsymbol{\nabla}\,\boldsymbol{\nabla\cdot}\overline{\boldsymbol{\gamma}}_i}{T}\Big) - L_{\mathrm{ee}}\boldsymbol{\nabla}\Big(\frac{-1}{T}\Big),$$

- **Structure of the matrix** L

$L = (L_{ij})_{i,j\in\mathfrak{S}\cup\{\mathrm{e}\}}$ symmetric positive semi-definite

$N(L) = \mathrm{Span}(1,\ldots,1,0)^t$

Thermochemistry of Fluid Mixtures (1)

- **Thermodynamics : $\mathcal{E}$, p, $\mathcal{S}$ are $C^\varkappa$ functions of $\mathsf{z}^{\mathrm{cl}} = (\rho_1, \dots, \rho_n, T)^t$ such that**

($\mathcal{T}_1$) The map $\mathsf{z}^{\mathrm{cl}} \to \mathsf{u}^{\mathrm{cl}}$ where $\mathsf{u}^{\mathrm{cl}} = (\rho_1, \dots, \rho_n, \mathcal{E})^t$ is a $C^\varkappa$ diffeomorphism from $\mathcal{O}_{\mathsf{z}^{\mathrm{cl}}} \subset (0,\infty)^{\mathsf{n}_\mathrm{s}+1}$ onto $\mathcal{O}_{\mathsf{u}^{\mathrm{cl}}}$

($\mathcal{T}_2$) Letting $g_i = \partial_{\rho_i}\mathcal{E} - T\,\partial_{\rho_i}\mathcal{S}$ we have the volumetric Gibbs relation $T\,d\mathcal{S} = -\sum_{i\in\mathfrak{S}} g_i\,d\rho_i + d\mathcal{E}$ and the constraint $\sum_{i\in\mathfrak{S}} \rho_i g_i = \mathcal{E} + p - T\mathcal{S}$

($\mathcal{T}_3$) The Hessian matrix $\widetilde{\partial}^2_{\mathsf{u}^{\mathrm{cl}}\mathsf{u}^{\mathrm{cl}}}\mathcal{S}$ is negative definite

($\mathcal{T}_4$) For any $(\mathsf{y}_1, \dots, \mathsf{y}_{\mathsf{n}_\mathrm{s}}, T) \in (0,\infty)^{\mathsf{n}_\mathrm{s}+1}$ with $\sum_{i\in\mathfrak{S}} \mathsf{y}_i = 1$ $\quad \exists \rho_\mathrm{m} > 0$ with $\mathsf{z}^{\mathrm{cl}}{}_\rho = (\rho\mathsf{y}_1, \dots, \rho\mathsf{y}_{\mathsf{n}_\mathrm{s}}, T)^t \in \mathcal{O}_{\mathsf{z}^{\mathrm{cl}}}$ for $0 < \rho < \rho_\mathrm{m}$ and

$$\lim_{\rho\to 0} \frac{\mathcal{E}(\mathsf{z}^{\mathrm{cl}}{}_\rho) - \mathcal{E}^{\mathrm{pg}}(\mathsf{z}^{\mathrm{cl}}{}_\rho)}{\rho} = \lim_{\rho\to 0} \frac{p(\mathsf{z}^{\mathrm{cl}}{}_\rho) - p^{\mathrm{pg}}(\mathsf{z}^{\mathrm{cl}}{}_\rho)}{\rho} = \lim_{\rho\to 0} \frac{\mathcal{S}(\mathsf{z}^{\mathrm{cl}}{}_\rho) - \mathcal{S}^{\mathrm{pg}}(\mathsf{z}^{\mathrm{cl}}{}_\rho)}{\rho} = 0$$

Thermochemistry of Fluid Mixtures (2)

- **Perfect gases thermodynamics in terms of $\mathsf{z}^{\mathrm{cl}} = (\rho_1, \dots, \rho_n, T)^t$**

$$p^{\mathrm{pg}} = RT \sum_{i \in \mathfrak{S}} \frac{\rho_i}{m_i} \qquad \mathcal{O}^{\mathrm{pg}}_{\mathsf{z}^{\mathrm{cl}}} = (0, \infty)^{\mathsf{n_s}+1}$$

$$\mathcal{E}^{\mathrm{pg}} = \sum_{i \in \mathfrak{S}} \rho_i e_i^{\mathrm{pg}} \qquad e_i^{\mathrm{pg}} = e_i^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} c_{\mathrm{v}i}^{\mathrm{pg}}(\theta)\, d\theta$$

$$\mathcal{S}^{\mathrm{pg}} = \sum_{i \in \mathfrak{S}} \rho_i \mathcal{S}_i^{\mathrm{pg}} \qquad \mathcal{S}_i^{\mathrm{pg}} = s_i^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} \frac{c_{\mathrm{v}i}^{\mathrm{pg}}(\theta)}{\theta}\, d\theta - \frac{RT}{m_i} \log \frac{\rho_i}{m_i \gamma^{\mathrm{st}}}$$

- **Natural assumptions**

(PG) e_i^{st}, s_i^{st} are constants $\quad m_i > 0,\ R > 0$ are positive constants

$c_{\mathrm{v}i}^{\mathrm{pg}}$ are $C^\infty\big([0,\infty), \mathbb{R}\big)$ with $0 < \underline{c}_{\mathrm{v}} \leqslant c_{\mathrm{v}i}^{\mathrm{pg}}(T) \leqslant \overline{c}_{\mathrm{v}} \quad T \geqslant 0,\ i \in \mathfrak{S}$

Thermochemistry of Fluid Mixtures (3)

- **Complex chemistry**

$$\sum_{i\in\mathfrak{S}} \nu^{\mathrm{f}}_{ij}\mathfrak{M}_i \rightleftarrows \sum_{i\in\mathfrak{S}} \nu^{\mathrm{b}}_{ij}\mathfrak{M}_i \qquad j \in \mathfrak{R} = \{1,\ldots,n^{\mathrm{r}}\}$$

- **Reduced chemical potential** $\boldsymbol{\mu_i = m_i g_i / RT}$

$$\nu^{\mathrm{f}}_j = \begin{pmatrix} \nu^{\mathrm{f}}_{1j} \\ \vdots \\ \nu^{\mathrm{f}}_{\mathsf{n}_{\mathrm{s}}j} \end{pmatrix} \qquad \nu^{\mathrm{b}}_j = \begin{pmatrix} \nu^{\mathrm{b}}_{1j} \\ \vdots \\ \nu^{\mathrm{b}}_{\mathsf{n}_{\mathrm{s}}j} \end{pmatrix} \qquad \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{\mathsf{n}_{\mathrm{s}}} \end{pmatrix} \qquad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{\mathsf{n}_{\mathrm{s}}} \end{pmatrix}$$

- **Production rates**

$$\nu_j = \nu^{\mathrm{b}}_j - \nu^{\mathrm{f}}_j \qquad \omega = \sum_{j\in\mathfrak{R}} \nu_j \tau_j \qquad \tau_j = \mathcal{K}_j \big(\exp\langle \nu^{\mathrm{f}}_j, \mu\rangle - \exp\langle \nu^{\mathrm{b}}_j, \mu\rangle\big)$$

Thermochemistry of Fluid Mixtures (4)

- **Entropy production due to chemistry**

$$-\sum_{i\in\mathfrak{S}} \frac{g_i m_i \omega_i}{T} = \sum_{j\in\mathfrak{R}} R\mathcal{K}_j \big(\langle \nu_j^{\mathrm{f}}, \mu\rangle - \langle \nu_j^{\mathrm{b}}, \mu\rangle\big)\big(\exp\langle \nu_j^{\mathrm{f}}, \mu\rangle - \exp\langle \nu_j^{\mathrm{b}}, \mu\rangle\big)$$

- **Reduced chemical potential and activity**

$$\mu_i^{\mathrm{pg}} = \mu_i^{\mathrm{u,pg}}(T) + \log \gamma_i^{\mathrm{pg}} \qquad \gamma_i^{\mathrm{pg}} = \frac{\rho_i^{\mathrm{pg}}}{m_i} = \frac{\mathsf{y}_i}{\nu^{\mathrm{pg}} m_i} \qquad \nu^{\mathrm{pg}} = \frac{RT}{p\,m}$$

$$\mathrm{a}_i = \exp\big(\mu_i - \mu_i^{\mathrm{u,pg}}\big) \qquad \widetilde{\mathrm{a}}_i = \exp\big(\mu_i - \mu_i^{\mathrm{pg}}\big) \qquad \mathrm{a}_i = \widetilde{\mathrm{a}}_i\ \gamma_i^{\mathrm{pg}}$$

- **Generalized mass action law**

$$\tau_j = \mathcal{K}_j^{\mathrm{f}} \prod_{i\in\mathfrak{S}} \mathrm{a}_i^{\nu_{ij}^{\mathrm{f}}} - \mathcal{K}_j^{\mathrm{b}} \prod_{i\in\mathfrak{S}} \mathrm{a}_i^{\nu_{ij}^{\mathrm{b}}}$$

Thermochemistry of Fluid Mixtures (5)

- **Atom and mass conservation**

$$\mathfrak{a}_l = \begin{pmatrix} \mathfrak{a}_{1l} \\ \vdots \\ \mathfrak{a}_{\mathsf{n_s}l} \end{pmatrix} \qquad l \in \mathfrak{A} = \{1, \dots, n^a\} \qquad \boldsymbol{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_{\mathsf{n_s}} \end{pmatrix} = \sum_{l \in \mathfrak{A}} \widetilde{m}_l \, \mathfrak{a}_l$$

$$\mathcal{R} = \mathrm{span}\{\ \nu_j,\ \ j \in \mathfrak{R}\ \} \qquad \mathcal{A} = \mathrm{span}\{\ \mathfrak{a}_l,\ \ l \in \mathfrak{A}\ \} \qquad \omega \in \mathcal{R} \qquad \boldsymbol{m} \in \mathcal{A}$$

$$\langle \nu_j, \mathfrak{a}_l \rangle = 0 \quad j \in \mathfrak{R}, l \in \mathfrak{A} \qquad \mathcal{R} \subset \mathcal{A}^{\perp} \qquad \langle \omega, \boldsymbol{m} \rangle = 0$$

- **Equilibrium**

$$\sum_{i \in \mathfrak{S}} \frac{g_i m_i \omega_i}{T} = 0 \iff \omega_i = 0 \ \ i \in \mathfrak{S} \iff \tau_j = 0 \ \ j \in \mathfrak{R} \iff \mu \in \mathcal{R}^{\perp}$$

Thermochemistry of Fluid Mixtures (6)

(C_1) We have $\nu^{\mathrm{f}}_{ij},\ \nu^{\mathrm{b}}_{ij},\ \mathfrak{a}_{il}\ \in \mathbb{N} \qquad i \in \mathfrak{S}\ ,\ j \in \mathfrak{R}\ ,\ l \in \mathfrak{A}$

Atom conservation

$$\langle \nu^{\mathrm{b}}_j, \mathfrak{a}_l \rangle - \langle \nu^{\mathrm{f}}_j, \mathfrak{a}_l \rangle = \langle \nu_j, \mathfrak{a}_l \rangle = 0, \qquad j \in \mathfrak{R}, \quad l \in \mathfrak{A}$$

(C_2) We have $\widetilde{m}_l > 0 \quad l \in \mathfrak{A}$ and

$$m_i = \sum_{l \in \mathfrak{A}} \widetilde{m}_l\, \mathfrak{a}_{il}, \qquad i \in \mathfrak{S}$$

(C_3) The rate constants $\mathcal{K}_j$ for $j \in \mathfrak{R}$, are C^∞ positive functions of $T > 0$