Shape Optimization of Polynomial Functionals under Uncertainties on the Right-Hand Side of the State Equation

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Congrès SMAI 2023,
Le Gosier, Guadeloupe.

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23 May 2023

Objectives

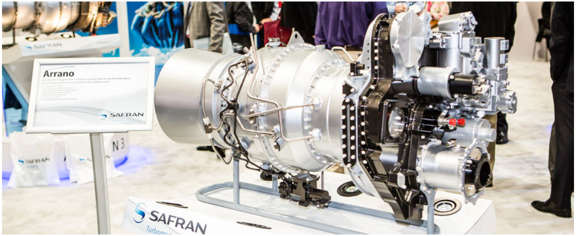


Figure – Photo of an Arrano engine. The gear crankcase is the black component on the right [1]

objectives of the optimization of a gear crankcase :

- Improve resistance with respect to **uncertain** mechanical loads (avoid high concentration of the von Mises stress)
- Reduce mass
- Assure airtightness
- Avoid regions occupied by other components

1. Picture of a Safran HE Arrano engine.
CAUCHI Philippe, *Turbomeca devient le motoriste exclusif du X4 d'Airbus Helicopters*, Info Aéro Québec, 18 February 2015. Available online at
https://infoaeroquebec.net/turbomeca-devient-le-motoriste-exclusif-du-x4-dairbus-helicopters/.
Consulted on the 5 May 2022.

The linear elasticity equations

We consider a domain Ω composed of an elastic material with Lamé parameters λ and μ.

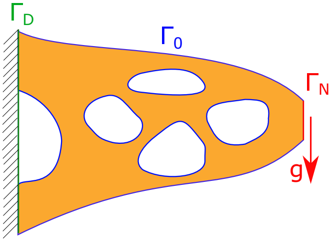
The structure is **clamped** on the portion Γ_D of its boundary, and a **mechanical load** g is applied on Γ_N. The free boundary is denoted Γ_0.

We denote $u_{\Omega,g}$ the **elastic displacement** of the structure, $\varepsilon\left(u_{\Omega,g}\right)$ the **symmetric gradient** of the displacement, and $\boldsymbol{\sigma}\left(u_{\Omega,g}\right)$ the **Cauchy stress tensor**, where :

$$\varepsilon\left(u_{\Omega,g}\right) = \frac{\nabla u_{\Omega,g} + \nabla u_{\Omega,g}^{\;T}}{2},$$
$$\boldsymbol{\sigma}\left(u_{\Omega,g}\right) = 2\mu\varepsilon\left(u_{\Omega,g}\right) + \lambda\mathbb{I}\left(\mathrm{div}(u_{\Omega,g})\right).$$

The displacement $u_{\Omega,g}$ is the solution of the following boundary values problem :

$$\left\{\begin{array}{rcll} -\mathrm{div}\,\boldsymbol{\sigma}\left(u_{\Omega,g}\right) & = & 0 & \text{in } \Omega, \\ \boldsymbol{\sigma}\left(u_{\Omega,g}\right)n & = & 0 & \text{on } \Gamma_0, \\ \boldsymbol{\sigma}\left(u_{\Omega,g}\right)n & = & g & \text{on } \Gamma_N, \\ u_{\Omega,g} & = & 0 & \text{on } \Gamma_D. \end{array}\right.$$



The von Mises stress

The **von Mises yield criterion** has been developed to assess the presence of plastic deformation in elasto-plastic materials.

According to the von Mises criterion, a structure made of an elasto-plastic material if, in each point of the structure, the **von Mises stress** s_{VM} is lower than the uniaxial yield stress σ_y.

The von Mises stress is defined as

$$s_{\mathrm{VM}}(\mathsf{x}) = \sqrt{\frac{2}{3}\left(\boldsymbol{\sigma}_{\mathrm{VM}}\left(\mathsf{u}_{\Omega,\mathbf{g}}(\mathsf{x})\right) : \boldsymbol{\sigma}_{\mathrm{VM}}\left(\mathsf{u}_{\Omega,\mathbf{g}}(\mathsf{x})\right)\right)}$$

where $\boldsymbol{\sigma}_{\mathrm{VM}}\left(\mathsf{u}_{\Omega,\mathbf{g}}\right) = \boldsymbol{\sigma}\left(\mathsf{u}_{\Omega,\mathbf{g}}\right) - \frac{1}{3}\mathbb{I}\,\mathrm{tr}\left(\boldsymbol{\sigma}\left(\mathsf{u}_{\Omega,\mathbf{g}}\right)\right)$ is the deviatoric part of the stress tensor.

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The square of the von Mises stress can be interpreted as a **density of distortion energy**, which is the fraction of elastic energy related to shear stresses.

The von Mises yield criterion can be used for **ductile materials** like aluminum or steel.

The L^m-norm of the von Mises stress

OBJECTIVE : design a structure avoiding high concentrations of von Mises stress.

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SOLUTION : approximate the L^∞-norm with a L^m-norm, for $m > 2$.

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EXAMPLE :
cantilever.

Let $\mathcal{O}_{adm}$ be a class of admissible domains.

Find $\Omega \in \mathcal{O}_{adm}$
minimizing $\|s_{\mathrm{VM}}\|_{L^m}$
under the constraint
$$\mathrm{Vol}(\Omega) \leq 2.0.$$

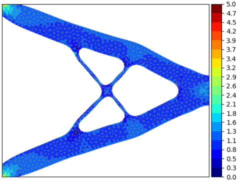


Figure – $m = 2$

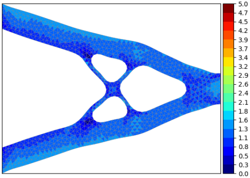
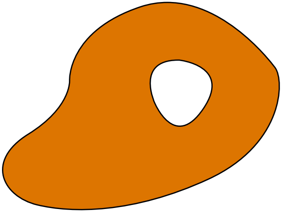


Figure – $m = 6$

	Volume	$\|s_{\mathrm{VM}}\|_2$	$\|s_{\mathrm{VM}}\|_6$
$m = 2$	2.0031	1.83893	1,7037
$m = 6$	2.00145	2,0073	1,6589

Moving boundary approach



Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, belonging to the class of admissible shapes $\mathcal{O}_{adm}$.

Let $\boldsymbol{\theta} \in \mathrm{W}^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ be a Lipschitz continuous vector field.

If $\|\boldsymbol{\theta}\|_{1,\infty} < 1$, the map $x \mapsto (\mathbb{I} + \boldsymbol{\theta})x$ is a Lipschitz homeomorphism with Lipschitz continuous inverse.

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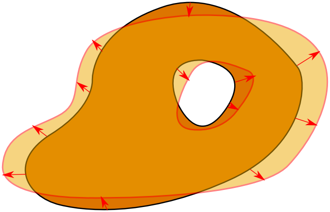

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The **deformed domain** $\Omega_{\boldsymbol{\theta}}$ is defined by applying the map $(\mathbb{I} + \boldsymbol{\theta})$ to each point of Ω :

$$\Omega_{\boldsymbol{\theta}} = (\mathbb{I} + \boldsymbol{\theta})\,\Omega.$$

Hadamard's shape derivative

Definition: Differentiable shape functional

A shape functional $J : \mathcal{O}_{adm} \to \mathbb{R}$ is **Fréchet differentiable** in $\Omega \in \mathcal{O}_{adm}$ if

1. $J(\Omega_{\boldsymbol{\theta}})$ is well defined for all $\boldsymbol{\theta} \in \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\|\boldsymbol{\theta}\|_{1,\infty} \leq 1$,
2. there exist a linear continuous map $J'(\Omega)(\cdot) : \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R}$ such that, for all $\theta \in \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$,

$$J(\Omega_{\boldsymbol{\theta}}) = J(\Omega) + J'(\Omega)(\boldsymbol{\theta}) + o(\|\boldsymbol{\theta}\|_{1,\infty}).$$

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Proposition: Hadamard's structure theorem [2]

Let Ω be a $\mathcal{C}^1$ domain in $\mathbb{R}^d$, and $J : \mathcal{O}_{adm} \to \mathbb{R}$ a differentiable shape functional.

The application $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d) \ni \boldsymbol{\theta} \to J'(\Omega)(\boldsymbol{\theta})$ is such that, if $\boldsymbol{\theta} \cdot \mathrm{n} = 0$ on $\partial\Omega$, then $J'(\Omega)(\boldsymbol{\theta}) = 0$.

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Thus, is is possible to define a **direction of descent** $\boldsymbol{\theta}_d$.

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A shape optimization problem under uncertainties

Let $\mathcal{O}_{adm}$ be a class of admissible domains in $\mathbb{R}^2$ or $\mathbb{R}^3$, $J : \mathcal{O}_{adm} \to \mathbb{R}$ an objective functional, and $P : (\Omega, \mathrm{u}) \mapsto P(\Omega, \mathrm{u}) \in \mathbb{R}$ a constraint functional taking as argument a domain $\Omega \in \mathcal{O}_{adm}$ and a function u defined on Ω.

We consider the following shape optimization problem :

Find $\Omega \in \mathcal{O}_{adm}$

minimizing $J(\Omega)$

under the constraint $P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}}) \leq M$

where the state $\mathrm{u}_{\Omega,\mathrm{g}}$ solves the linear elasticity equation :

$$\begin{cases} -\mathrm{div}\boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}\right) = 0 & \text{in } \Omega, \\ \boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}\right)\mathrm{n} = 0 & \text{on } \Gamma_0, \\ \boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}\right)\mathrm{n} = \mathrm{g} & \text{on } \Gamma_\mathrm{N}, \\ \mathrm{u}_{\Omega,\mathrm{g}} = 0 & \text{on } \Gamma_\mathrm{D}. \end{cases}$$

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We consider the mechanical load g to be a **random variable** belonging to a suitable Bochner space.

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Since the term $P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}})$ is a random variable, we have to express the constraint as a deterministic quantity.

Here, we consider its **expected value**.

Multilinear functionals and their shape derivatives

We focus on the case of **shape-differentiable** functionals $P(\cdot,\cdot)$ with a particular structure.

Let $P_\Omega : \underbrace{\mathrm{W}^{1,m}(\Omega) \times \cdots \mathrm{W}^{1,m}(\Omega)}_{m} \to \mathbb{R}$ be an m-multilinear functional such that :

1. P_Ω is continuous : $P_\Omega(\mathrm{u}_1,\ldots,\mathrm{u}_m) \leq K \|\mathrm{u}_1\|_{\mathrm{W}^{1,m}(\Omega)} \cdots \|\mathrm{u}_1\|_{\mathrm{W}^{1,m}(\Omega)}$;
2. $P(\Omega, \mathrm{u}_{\Omega,\mathbf{g}}) = P_\Omega\left(\mathrm{u}_{\Omega,\mathbf{g}},\ldots,\mathrm{u}_{\Omega,\mathbf{g}}\right)$ if $\mathrm{u}_{\Omega,\mathbf{g}} \in \mathrm{W}^{1,m}(\Omega)$;
3. P_Ω can be written as follows :

$$P_\Omega(\mathrm{u}_1,\ldots,\mathrm{u}_m) = \int_\Omega (q_1(\mathrm{u}_1,\ldots,\mathrm{u}_m) + q_2(\nabla\mathrm{u}_1,\ldots,\nabla\mathrm{u}_m))\,\mathrm{d}x.$$

Such functionals are **shape-differentiable**, and the computation of the derivative requires an **adjoint state**.

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Examples of such functionals include :

- the mechanical compliance

$$C(\Omega,\mathrm{u}_{\Omega,\mathrm{g}}) = C_\Omega(\mathrm{u}_{\Omega,\mathrm{g}},\mathrm{u}_{\Omega,\mathrm{g}}) = \int_\Omega \boldsymbol{\sigma}(\mathrm{u}) : \boldsymbol{\varepsilon}(\mathrm{u})\,\mathrm{dx};$$

- the m-th power of the L^m-norm of the von Mises stress

$$G(\Omega,\mathrm{u}_{\Omega,\mathrm{g}}) = G_\Omega(\underbrace{\mathrm{u}_{\Omega,\mathrm{g}},\ldots,\mathrm{u}_{\Omega,\mathrm{g}}}_{m}) = \int_\Omega \left(\boldsymbol{\sigma}_{\mathrm{VM}}(\mathrm{u}) : \boldsymbol{\sigma}_{\mathrm{VM}}(\mathrm{u})\right)^{m/2}\mathrm{dx}.$$

Modeling the uncertainties

Let the mechanical load g and the displacement $\mathrm{u}_{\Omega,\mathrm{g}}$ be random variables with respect to the probability space $(\mathfrak{O}, \mathcal{F}, \mathbb{P})$.

For $\mathbb{E}\left[P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}})\right] = \mathbb{E}\left[P_\Omega\left(\mathrm{u}_{\Omega,\mathrm{g}}, \dots, \mathrm{u}_{\Omega,\mathrm{g}}\right)\right]$ to be well-defined, we have to assume that $\mathrm{u}_{\Omega,\mathrm{g}}$ belongs to the Bochner space $\mathrm{L}^m\left(\mathrm{W}^{m,1}(\Omega); \mathbb{P}\right)$.

PROBLEM : how can we compute and differentiate $\mathbb{E}\left[P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}})\right]$?

3. M. Dambrine, C. Dapogny, H. Harbrecht. "Shape Optimization for Quadratic Functionals and States with Random Right-Hand Sides". *SIAM Journal on Control and Optimization* 53.5 (2015) : 3081-3103.

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IDEA : extend the approach of (Dambrine, Dapogny, Harbrecht, 2015)[3] to multilinear functionals.

The method we propose allows to differentiate the inequality constraint, and relies on the following steps :

- identification of the deterministic **correlation tensor** $\mathrm{Cor}(g, \ldots, g) = \mathbb{E}\left[g \otimes \ldots \otimes g\right]$;
- decomposition of the correlation tensor, assuming that g is a sum of independent variables ;
- computation of the shape derivative for each term of the decomposition.

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Tensor product of multiple Banach spaces (1)

Let us consider a vector spaces X and a positive integer $m \geq 2$. We denote $\hat{\mathfrak{P}}_m(X^m)$ the space of all m-multilinear forms on X^m.

Definition: Tensor product between vector spaces

For $(x_1, \ldots, x_m) \in X^m$, the **tensor product** $x_1 \otimes \ldots \otimes x_m$, also written as $\bigotimes_{i=1}^m x_i$, is a real valued linear application defined on $\hat{\mathfrak{P}}_m(X^m)$ such that, for all $P_m \in \hat{\mathfrak{P}}_m(X^m)$,

$$\left(\bigotimes_{i=1}^{m} x_i\right)(P_m) = P_m(x_1, \ldots, x_m).$$

The m-tensor product of the vector space X is defined as :

$$\bigotimes_{i=1}^{m} X = \text{span}\left\{\bigotimes_{i=1}^{m} x_i \quad \text{such that} \quad x_i \in X \quad \forall i = 1 \ldots m\right\}.$$

Tensor product of multiple Banach spaces (2)

Definition: Projective norm

Let X be a Banach space provided with the norm $\|\cdot\|_X$. By definition, every element u of $\bigotimes_{i=1}^m X$ can be written as a finite sum of tensor products :
$u = \sum_{j=1}^N x_1^j \otimes \ldots \otimes x_m^j$, but such representation is not necessarily unique. Let $\pi(\cdot)$ be the following real mapping, defined on $\bigotimes_{i=1}^m X$:

$$\pi(u) = \inf \left\{ \sum_{j=1}^N \left(\prod_{i=1}^m \left\| x_i^j \right\|_X \right) \quad : \quad u = \sum_{j=1}^N x_1^j \otimes \ldots \otimes x_m^j \right\}.$$

The function $\pi(\cdot)$ is called **projective norm**.

Definition: Projective product space

The completion of the normed vector space $\bigotimes_{i=1}^m X$ with respect to the projective norm $\pi(\cdot)$ is the **projective product space**, which is a Banach space and is denoted as $\widehat{\bigotimes}_{\pi,i=1}^m X$.

The correlation tensor

Let $(\mathfrak{O}, \mathcal{F}, \mathbb{P})$ be a probability space, and X a Banach space. Let us consider the Bochner spaces $\mathrm{L}^m(X;\mathbb{P})$, and m random variables $x_1, \ldots x_m \in \mathrm{L}^m(X;\mathbb{P})$.

Definition: Correlation tensor

The **correlation operator** $\mathrm{Cor}_m : (\mathrm{L}^m(X;\mathbb{P}))^m \to \widehat{\bigotimes}_{\pi,i=1}^m X$ maps a vector of m random variables to their correlation tensor :

$$\mathrm{Cor}_m(x_1, \ldots, x_m) = \mathbb{E}\left[x_1 \otimes \ldots \otimes x_m\right].$$

The term $\mathrm{Cor}_m(x_1, \ldots, x_m)$ is the **correlation tensor** relative to the variables $x_1, \ldots, x_m$

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The term $\mathrm{Cor}_m(x_1, \dots, x_m)$ is the **correlation tensor** relative to the variables $x_1, \dots, x_m$

Proposition: Expectation of a multilinear operator

Let $P_m : X^m \to \mathbb{R}$ a bounded m-multilinear operator. Then, there exists a unique bounded, real-valued, linear operator $\widehat{P_m}$ defined on $\widehat{\bigotimes}_{\pi,i=1}^m X$ such that these three statements hold true for all $(x_1, \dots, x_m) \in (\mathrm{L}^m(X;\mathbb{P}))^m$:

1. $P_m(x_1, \dots, x_m) \in \mathrm{L}^1 C(\mathfrak{O}, \mathbb{P})$,
2. $P_m(x_1(\omega), \dots, x_m(\omega)) = \widehat{P_m}\left(x_1(\omega) \otimes \dots x_n(\omega)\right)$, for almost all $\omega \in \mathfrak{O}$,
3. $\mathbb{E}\left[P_m(x_1, \dots, x_m)\right] = \widehat{P_m}\left(\mathrm{Cor}_m(x_1, \dots, x_m)\right)$.

Optimization problem in elasticity under uncertainties

We can now come back to the initial shape optimization problem under uncertainties.

$$
\left|
\begin{array}{l}
\text{Find } \Omega \in \mathcal{O}_{adm} \\
\text{minimizing } J(\Omega) \\
\text{under the constraint } \mathbb{E}\left[P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}})\right] \leq M \\
\text{where the state } \mathrm{u}_{\Omega,\mathrm{g}}(\omega) \text{ solves the} \\
\text{ linear elasticity equation :} \\
\left\{
\begin{array}{rcll}
-\mathrm{div}\boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}(\omega)\right) & = & 0 & \text{in } \Omega, \\
\boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}(\omega)\right)\mathrm{n} & = & 0 & \text{on } \Gamma_0, \\
\boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}(\omega)\right)\mathrm{n} & = & \mathrm{g}(\omega) & \text{on } \Gamma_{\mathrm{N}}, \\
\mathrm{u}_{\Omega,\mathrm{g}}(\omega) & = & 0 & \text{on } \Gamma_{\mathrm{D}}.
\end{array}
\right.
\end{array}
\right.
$$

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\left|
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\mathrm{u}_{\Omega,\mathrm{g}}(\omega) & = & 0 & \text{on } \Gamma_{\mathrm{D}}.
\end{array}
\right.
\end{array}
\right.
$$

We consider $\mathrm{g} \in \mathrm{L}^m\left(\mathrm{L}^2(\Gamma_{\mathrm{N}}); \mathbb{P}\right)$ to be a finite sum of random variables as in :

$$
\mathrm{g}(\omega) = \sum_{k=1}^{N} \mathrm{g}_k\, \xi_k(\omega), \tag{1}
$$

where, all $\mathrm{g}_k \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ are regular mechanical loads, and $\xi_k \in \mathrm{L}^m(\mathbb{R}; \mathbb{P})$ are **mutually independent** real-valued random variables.

Shape derivative under uncertainties (1)

Proposition: Shape derivative of the expectation of a multilinear functional (1)

Let Ω be a $\mathcal{C}^1$ domain belonging to the interior of $\mathcal{O}_{adm}$. Moreover, let us consider that $\mathrm{g} \in \mathrm{L}^m\left(\mathrm{L}^2(\Gamma_\mathrm{N}); \mathbb{P}\right)$ can be decomposed as in (1), where the N real random variables $\xi_i \in \mathrm{L}^m\left(\mathbb{R}; \mathbb{P}\right)$ are mutually independent.
Then, we can write the shape derivative of the objective in Ω as follows :

$$\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathbb{E}\left[P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}})\right](\boldsymbol{\theta}) = -\sum_{j=1}^{N}\int_{\Gamma_0}(\boldsymbol{\theta}\cdot\mathrm{n})\,\left(\boldsymbol{\sigma}\left(\mathrm{u}_j\right):\boldsymbol{\varepsilon}\left(\mathrm{w}_j\right)\right)\,\mathrm{ds}$$

$$+\sum_{\vec{\boldsymbol{k}}\in\mathcal{A}_{(\mathbf{1},m),N}}\left(\alpha(\vec{\boldsymbol{k}})\int_{\Gamma_0}(\boldsymbol{\theta}\cdot\mathrm{n})\,\left(q_1(\mathrm{u}_{k_1},\ldots,\mathrm{u}_{k_m})+q_2(\nabla\mathrm{u}_{k_1},\ldots,\nabla\mathrm{u}_{k_m})\right)\,\mathrm{ds}\right).$$

...

Shape derivative under uncertainties (2)

Proposition: Shape derivative of the expectation of a multilinear functional (2)

...

The N states $\mathrm{u}_1, \ldots, \mathrm{u}_N$ solve the state equation for $\mathrm{g}_1, \ldots, \mathrm{g}_N$ respectively, while the N adjoint states $\mathrm{w}_1, \ldots, \mathrm{w}_N$ solve the following adjoint problems :

$$\begin{cases} -\mathrm{div}\boldsymbol{\sigma}\left(\mathrm{w}_j\right) = & \sum_{i=1}^{m} \sum_{\vec{\boldsymbol{k}} \in \mathcal{A}^{i,j}_{(\mathbf{1},m),N}} \alpha(\vec{\boldsymbol{k}}) \left(\frac{\partial q_1}{\partial \mathrm{v}_i}(\mathrm{u}_{k1}, \ldots, \mathrm{u}_{km}) \right. \\ & \left. - \; \mathrm{div} \frac{\partial q_2}{\partial \mathrm{V}_i}(\nabla \mathrm{u}_{k1}, \ldots, \nabla \mathrm{u}_{km}) \right) & \text{in } \Omega, \\ \boldsymbol{\sigma}\left(\mathrm{w}_j\right)\mathrm{n} = & \sum_{i=1}^{m} \sum_{\vec{\boldsymbol{k}} \in \mathcal{A}^{i,j}_{(\mathbf{1},m),N}} \alpha(\vec{\boldsymbol{k}}) \left(\frac{\partial q_2}{\partial \mathrm{V}_i}(\nabla \mathrm{u}_{k1}, \ldots, \nabla \mathrm{u}_{km}) \right)^T \mathrm{n} & \text{on } \Gamma_0 \cup \Gamma_{\mathrm{N}}, \\ \mathrm{w}_j = & 0 & \text{on } \Gamma_{\mathrm{D}}. \end{cases}$$

- $\mathcal{A}_{(\mathbf{1},m),N} = \{1, \ldots, N\}^m$;
- $\mathcal{A}^{i,j}_{(\mathbf{1},m),N} = \left\{ \vec{\boldsymbol{k}} \in \mathcal{A}_{(\mathbf{1},m),N} \text{ such that } k_i = i \right\} \subset \mathcal{A}_{(\mathbf{1},m),N}$;
- for $\xi_1, \ldots, \xi_m \in \mathrm{L}^m(\mathbb{R}; \mathbb{P})$, we denote $\mu_{i,j} = \mathbb{E}\left[\xi_j^i\right]$;
- finally, for $\vec{\boldsymbol{k}} = (k_1, \ldots, k_m) \in \mathcal{A}_{(\mathbf{1},m),N}$, we denote $\alpha(\vec{\boldsymbol{k}}) = \prod_{j=1}^N \left(\mathbb{E}\left[\xi_j^{C^j_{\vec{\boldsymbol{k}}}} \right] \right) = \prod_{j=1}^N \mu_{C^j_{\vec{\boldsymbol{k}}},j}$.

Complexity

We denote $\mathcal{T}(P_\Omega(m), N)$ the number of terms appearing in the expression of the shape derivative of a m-multilinear functional, when the load $g \in \mathrm{L}^m\left(\mathrm{L}^2(\Gamma_\mathrm{N}); \mathbb{P}\right)$ is decomposed in N mutually independent random variables.

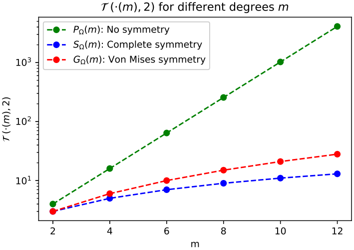


Figure – Case of $N = 2$ random variables

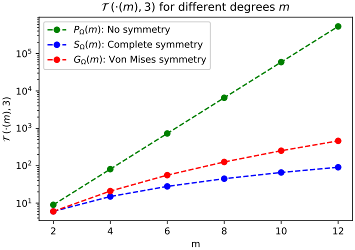


Figure – Case of $N = 3$ random variables

$$\mathcal{T}(P_\Omega(m), N) = N^m, \qquad \mathcal{T}(S_\Omega(m), N) = \binom{N+m-1}{m},$$

$$\mathcal{T}(G_\Omega(m), N) = \binom{\frac{N(N+1)}{2} + \frac{m}{2} - 1}{\frac{m}{2}}.$$

3D optimization under von Mises constraint

We consider a set of 3D admissible shapes $\mathcal{O}_{adm}$, sharing the portions $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$, a space of events $\mathcal{O}$, and a probability measure $\mathbb{P}$. We suppose g to be the random mechanical load applied to $\Gamma_{\rm N}$.

3D optimization under von Mises constraint

We consider a set of 3D admissible shapes $\mathcal{O}_{adm}$, sharing the portions Γ_D and Γ_N, a space of events $\mathcal{O}$, and a probability measure $\mathbb{P}$. We suppose g to be the random mechanical load applied to Γ_N.

We aim to solve the following shape optimization problem.

Find $\Omega \in \mathcal{O}_{adm}$

minimizing $\Omega \mapsto \mathrm{Vol}(\Omega)$,

such that, for all $\omega \in \mathcal{O}$,

the state $u_{\Omega,g}(\omega) \in \left[\mathrm{H}^1(\Omega)\right]^d$ solves :

$$\begin{cases} -\mathrm{div}\,\boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) = 0 & \text{in } \Omega, \\ \boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) n = g(\omega) & \text{on } \Gamma_N, \\ \boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) n = 0 & \text{on } \Gamma_0, \\ u_{\Omega,g}(\omega) = 0 & \text{on } \Gamma_D. \end{cases}$$

and the following constraint holds :

$$\mathbb{E}\left[G_6(\Omega, u_{\Omega,g})\right] \leq M_0^6,$$

where $G_6(\Omega, u_{\Omega,g}) = \|s_{\mathrm{VM}}\|_6^6$

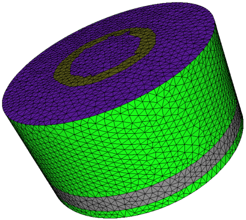
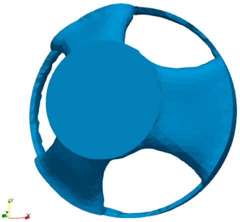
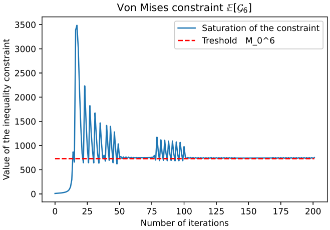


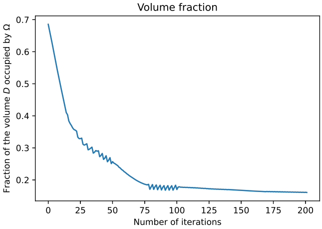
Figure – Representation of the structure to be optimized. The surface Γ_D is the thin grey stripe on the lateral surface, while Γ_N is the ring-shaped portion of the upper surface marked in yellow.

Numerical result (Isotropic load)

Load $g(\omega)$	$g_x\, X(\omega) + g_y\, Y(\omega)$
Variance of X	2.5
Variance of Y	2.5
Treshold M_0	3.0
Iterations of optimization algorithm	200
Time of execution	129 min.
Final volumic fraction $\mathrm{Vol}(\Omega)/\mathrm{Vol}(D)$	0.1608
Normalized saturation of the constraint $(\mathbb{E}[\mathcal{G}_6] - M_0^6)/M_0^6$	0.03002

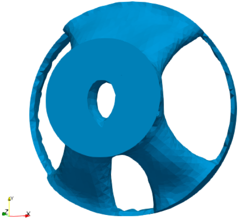


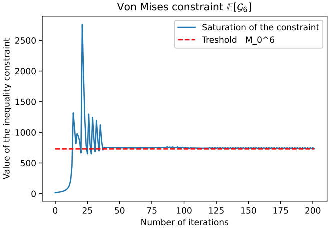


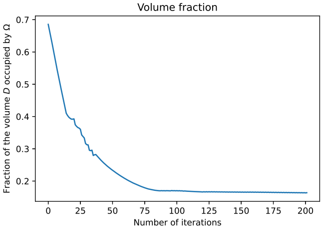


Numerical result (Anisotropic load)

Load $g(\omega)$ Variance of X Variance of Y	$g_x\, X(\omega) + g_y\, Y(\omega)$ 1.0 4.0
Treshold M_0	3.0
Iterations of optimization algorithm Time of execution	200 148 min.
Final volumic fraction $\mathrm{Vol}(\Omega)/\mathrm{Vol}(D)$ Normalized saturation of the constraint $(\mathbb{E}[\mathcal{G}_6] - M_0^6)/M_0^6$	0.164 0.005351







Summary and perspectives

Summary The approach to shape optimization adopted in this presentation

- can be applied to continuous functionals that can be written as a polynomial expression of degree m of the state of a shape optimization problem ;
- allows to model for boundary value problems with a random right-hand side, without any assumption on the size of the uncertainties ;
- provides a deterministic expression for the shape derivative, which depends on the first m moments of the random variables modeling the uncertain load ;
- has been applied to minimize the volume of a structure under constraints on the L^6-norm of the von Mises stress, or under constraints on the expectation of a quadratic functional.

A paper about this subject has been submitted to the journal *Numerische Mathematik*.

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A paper about this subject has been submitted to the journal *Numerische Mathematik*.

Perspectives Directions of further investigation include :

- application of tensor decomposition techniques to the *correlation tensor* in order to accelerate the algorithm ;
- introduction of the airtightness constraint ;
- industrialization of the code and coupling with the tools in use at Safran HE ;

Constraints on the worst-case scenario

Find $\Omega \in \mathcal{O}_{adm}$

minimizing $\Omega \mapsto \mathrm{Vol}(\Omega)$,

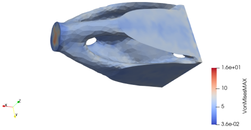
such that, for all $\mathrm{g} \in \mathcal{G}$,

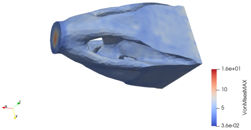
the state $\mathrm{u}_{\Omega,\mathrm{g}} \in \left[\mathrm{H}^1(\Omega)\right]^d$ solves :

$$\begin{cases} -\mathrm{div}\, \boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}\right) = 0 & \text{in } \Omega, \\ \boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}\right)\mathrm{n} = \mathrm{g}(\omega) & \text{on } \Gamma_{\mathrm{N}}, \\ \boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}\right)\mathrm{n} = 0 & \text{on } \Gamma_0, \\ \mathrm{u}_{\Omega,\mathrm{g}} = 0 & \text{on } \Gamma_{\mathrm{D}}. \end{cases}$$

and the following constraint holds :

$$\sup_{\mathrm{g}\in\mathcal{G}} P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}}) \leq M_0,$$





where :

- $P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}}) = P_\Omega(\mathrm{u}_{\Omega,\mathrm{g}}, \ldots, \mathrm{u}_{\Omega,\mathrm{g}})$;
- $\mathrm{g} \mapsto P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}})$ is convex ;
- $P_\Omega(\cdot, \ldots, \cdot)$ is a m-multilinear functional ;
- $\mathcal{G}$ is a compact subset of a finite-dimensional Banach space.

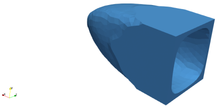
Constraints on the probability of exceeding a treshold

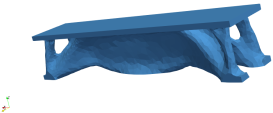
Find $\Omega \in \mathcal{O}_{adm}$
minimizing $\Omega \mapsto \text{Vol}(\Omega)$,
such that, for all $\omega \in \mathcal{O}$,
the state $u_{\Omega,g} \in \left[H^1(\Omega)\right]^d$ solves :

$$\begin{cases} -\text{div}\, \boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) = 0 & \text{in } \Omega, \\ \boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) n = g(\omega) & \text{on } \Gamma_N, \\ \boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) n = 0 & \text{on } \Gamma_0, \\ u_{\Omega,g}(\omega) = 0 & \text{on } \Gamma_D. \end{cases}$$

and the following constraint holds :

$$\mathbb{P}\left[P(\Omega, u_{\Omega,g}) \geq s\right] \leq M_0,$$





REMARK : It is necessary to assume that g follows a given probability distribution.

Thank you for your attention

CONTACTS

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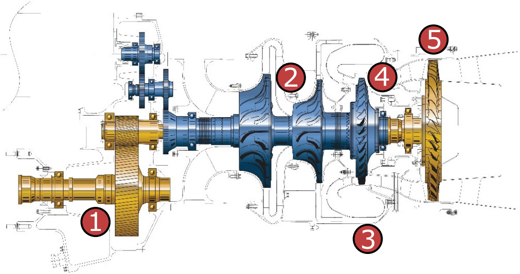
The components of the engine

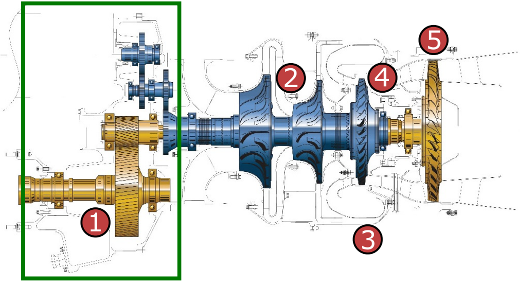


1. Gear box
2. Compressors
3. Combustion chamber
4. Gas generator turbine
5. Power turbine

Figure – Scheme of the Safran HE Arrano engine [3]

3. Adapted from Scheme of the Safran HE Arrano engine.
LAGARDE Philippe, *Le futur de la propulsion d'hélicoptère* in *La Jaune et la Rouge*, magazine N° 767, September 2022 (in French). Available online at
https://www.lajauneetlarouge.com/le-futur-de-la-propulsion-dhelicoptere/

The components of the engine



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☐ Gear crankcase

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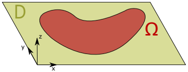
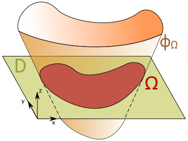
Level-set function



Let $D \subset \mathbb{R}^d$ be an open and bounded domain in $\mathbb{R}^d$.

Level-set function

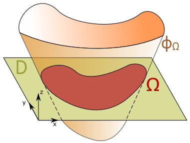


Let $D \subset \mathbb{R}^d$ be an open and bounded domain in $\mathbb{R}^d$.

The shape $\Omega \subset D \subset \mathbb{R}^d$ is parametrized by a continuous **level set** function $\phi_\Omega : \mathbb{R}^d \to \mathbb{R}$ such that :

$$\begin{cases} \phi_\Omega(x) > 0 & \text{if } x \notin \bar{\Omega}, \\ \phi_\Omega(x) = 0 & \text{if } x \in \partial\Omega, \\ \phi_\Omega(x) < 0 & \text{if } x \in \mathring{\Omega}. \end{cases}$$

Level-set function



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Let $V(\cdot;\cdot) : [0,T] \times D \to \mathbb{R}^d$ be a smooth Lagrangian velocity field defined on D for a time interval $[0,T]$.

It is possible to associate a direction of descent $\boldsymbol{\theta}$ with a velocity field V_θ using a suitable Hilbertian extension procedure.

In this case, the level-set function evolves according to the **advection equation**

$$\frac{\partial \phi}{\partial t}(t;\mathrm{x}) + V_{\boldsymbol{\theta}}(t;\mathrm{x}) \cdot \nabla\phi(t;\mathrm{x}) = 0.$$

Proposition: Shape derivative of a differentiable multilinear functional

Let Ω be a $\mathcal{C}^1$ domain belonging to the interior of $\mathcal{O}_{adm}$, and $P(\cdot,\cdot)$ a continuous shape functional that respects the structure outlined earlier.
Then, its shape derivative can be written as :

$$\frac{\mathrm{d}}{\mathrm{d}\Omega} P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}})(\boldsymbol{\theta}) = \int_{\Gamma_0} (\boldsymbol{\theta}\cdot \mathrm{n}) \big(q_1(\mathrm{u}_{\Omega,\mathrm{g}}, \dots, \mathrm{u}_{\Omega,\mathrm{g}}) + q_2(\nabla \mathrm{u}_{\Omega,\mathrm{g}}, \dots, \nabla \mathrm{u}_{\Omega,\mathrm{g}}) - \boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}\right) : \nabla \mathrm{w}_{\Omega,\mathrm{g}} \big) \, \mathrm{d}s,$$

where the adjoint state $\mathrm{w}_{\Omega,\mathrm{g}}$ is the solution of the following problem :

$$\begin{cases} -\mathrm{div}\, \boldsymbol{\sigma}\left(\mathrm{w}_{\Omega,\mathrm{g}}\right) = \sum_{i=1}^m \frac{\partial q_1}{\partial \mathrm{v}_i}(\mathrm{u}_{\Omega,\mathrm{g}}, \dots, \mathrm{u}_{\Omega,\mathrm{g}}) - \mathrm{div} \frac{\partial q_2}{\partial \mathrm{V}_i}(\nabla \mathrm{u}_{\Omega,\mathrm{g}}, \dots, \nabla \mathrm{u}_{\Omega,\mathrm{g}}) & \text{in } \Omega \\ \boldsymbol{\sigma}\left(\mathrm{w}_{\Omega,\mathrm{g}}\right)\mathrm{n} = \sum_{i=1}^m \frac{\partial q_2}{\partial \mathrm{V}_i}(\nabla \mathrm{u}_{\Omega,\mathrm{g}}, \dots, \nabla \mathrm{u}_{\Omega,\mathrm{g}})\,\mathrm{n} & \text{on } \Gamma_0 \cap \Gamma_{\mathrm{N}}, \\ \mathrm{w}_{\Omega,\mathrm{g}} = 0 & \text{on } \Gamma_{\mathrm{D}}. \end{cases}$$

Computation of the shape derivative

Proposition: Shape derivative of a differentiable multilinear functional

Let Ω be a $\mathcal{C}^1$ domain belonging to the interior of $\mathcal{O}_{adm}$, and $P(\cdot,\cdot)$ a continuous shape functional that respects the structure outlined earlier.
Then, its shape derivative can be written as :

$$\frac{\mathrm{d}}{\mathrm{d}\Omega} P(\Omega, \mathrm{u}_{\Omega,\mathrm{g}})(\boldsymbol{\theta}) = \int_{\Gamma_0} (\boldsymbol{\theta}\cdot \mathrm{n}) \Big(q_1(\mathrm{u}_{\Omega,\mathrm{g}}, \dots, \mathrm{u}_{\Omega,\mathrm{g}}) + q_2(\nabla \mathrm{u}_{\Omega,\mathrm{g}}, \dots, \nabla \mathrm{u}_{\Omega,\mathrm{g}}) - \boldsymbol{\sigma}\left(\mathrm{u}_{\Omega,\mathrm{g}}\right) : \nabla \mathrm{w}_{\Omega,\mathrm{g}} \Big)\, \mathrm{d}s,$$

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This result can be proven using Céa's fast derivation method, or by computing the volumetric form of the Eulerian derivative and applying Hadamard's structure theorem.

Linearization of continuous multilinear functionals

Proposition: Linearization of bounded multilinear functionals

Let us consider a real-valued, bounded, multilinear functional $P_m : X^m \to \mathbb{R}$.
For any Banach space B, we denote B^* its topological dual.
Then, there exists a unique linear functional $\widehat{P_m} : \widehat{\bigotimes}_{\pi,i=1}^{m} X \to \mathbb{R}$ such that :

1. the functional $\widehat{P_m}$ is continuous, and $\left\|\widehat{P_m}\right\|_{\mathrm{OP}} = \sup_{\|x_i\|_X=1\,\forall i} |P_m(x_1,\ldots,x_m)| = \|P_m\|_{\mathrm{OP}}$;
2. for all $(x_1,\ldots,x_m) \in X^m$, $\widehat{P_m}\left(\bigotimes_{i=1}^m x_i\right) = P_m(x_1,\ldots,x_m)$.

Moreover, the correspondence $P_m \leftrightarrow \widehat{P_m}$ is an isometric isomorphism between $\left(\widehat{\bigotimes}_{\pi,i=1}^{m} X\right)^*$ and the Banach spaces of the continuous m-multilinear functionals $X^m \to \mathbb{R}$.

Proof. The proof is an application of the Hahn-Banach extension theorem.

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Proof. The proof is an application of the Hahn-Banach extension theorem.

- Consider a linear functional $\widetilde{P_m} : \bigotimes_{i=1}^m X \to \mathbb{R}$ such that $\widetilde{P_m}\left(\sum_{j=1}^n \bigotimes_{i=1}^m x_i^n\right) = \sum_{j=1}^n P_m(x_1^j,\ldots,x_m^j)$.
- $\widetilde{P_m}$ is well defined and continuous on $\bigotimes_{i=1}^m X$ with respect to the projective norm.
- By the Hahn-Banach theorem, there exists a continuous operator $\widehat{P_m} : \widehat{\bigotimes}^{m}_{\pi,i=1} X \to \mathbb{R}$ that extends $\widetilde{P_m}$ and has the same norm.
- Since $\bigotimes_{i=1}^m X$ is dense in $\widehat{\bigotimes}^{m}_{\pi,i=1} X$, such extension is unique.

we introduce the following notation :

- $\mathcal{A}_{(1,m),N} = \{1, \dots, N\}^m$ is the set of all m-uples whose elements are integers between 1 and N;
- $\mathcal{A}^{i,j}_{(1,m),N} = \left\{ \vec{\boldsymbol{k}} \in \mathcal{A}_{(1,m),N} \text{ such that } k_i = i \right\} \subset \mathcal{A}_{(1,m),N}$ is the subset of all m-uples in $\mathcal{A}_{(1,m),N}$ whose i-th element is equal to j;
- given N real random variables $\xi_1, \dots, \xi_m$ belonging to the Bochner space $\mathrm{L}^m(\mathbb{R};\mathbb{P})$ and a m-uple $\vec{\boldsymbol{k}} = (k_1, \dots, k_m) \in \mathcal{A}_{(1,m),N}$, we denote $\mu_{i,j}$ the i-th moment of the random variable ξ_j : $\mu_{i,j} = \mathbb{E}\left[\xi_j^i\right]$;
- finally, we denote $\alpha(\vec{\boldsymbol{k}})$ the following quantity :

$$\alpha(\vec{\boldsymbol{k}}) = \alpha(k_1, \dots, k_m) = \prod_{j=1}^{N} \left(\mathbb{E}\left[\xi_j^{C^j_{\vec{\boldsymbol{k}}}}\right] \right) = \prod_{j=1}^{N} \mu_{C^j_{\vec{\boldsymbol{k}}},j}.$$

Sketch of the proof

Proof.

- Expression of $\mathbb{E}\left[P(\Omega, \mathrm{u}_{\Omega,\mathbf{g}})\right]$ in terms of the correlation tensor :

$$\mathbb{E}\left[P(\Omega, \mathrm{u}_{\Omega,\mathbf{g}})\right] = \mathbb{E}\left[P_\Omega\left(\mathrm{u}_{\Omega,\mathbf{g}}, \ldots, \mathrm{u}_{\Omega,\mathbf{g}}\right)\right] = \widehat{P_\Omega}\left(\mathrm{Cor}(\mathrm{u}_{\Omega,\mathbf{g}}, \ldots, \mathrm{u}_{\Omega,\mathbf{g}})\right).$$

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- Decomposition of the correlation tensor using the linearity of the functional $\widehat{P_\Omega}$:

$$\mathrm{Cor}(\mathrm{u}_{\Omega,\mathbf{g}}, \ldots, \mathrm{u}_{\Omega,\mathbf{g}}) = \sum_{\vec{\boldsymbol{k}} \in \mathcal{A}_{(\mathbf{1},m),N}} \left(\alpha(\vec{\boldsymbol{k}})\left(\mathrm{u}_{k_\mathbf{1}} \otimes \ldots \otimes \mathrm{u}_{k_m}\right)\right);$$

therefore

$$\widehat{P_\Omega}\left(\mathrm{Cor}(\mathrm{u}_{\Omega,\mathbf{g}}, \ldots, \mathrm{u}_{\Omega,\mathbf{g}})\right) = \widehat{P_\Omega}\left(\sum_{\vec{\boldsymbol{k}} \in \mathcal{A}_{(\mathbf{1},m),N}} \left(\alpha(\vec{\boldsymbol{k}})\left(\mathrm{u}_{k_\mathbf{1}} \otimes \ldots \otimes \mathrm{u}_{k_m}\right)\right)\right) =$$

$$\sum_{\vec{\boldsymbol{k}} \in \mathcal{A}_{(\mathbf{1},m),N}} \alpha(\vec{\boldsymbol{k}})\widehat{P_\Omega}\left(\mathrm{u}_{k_\mathbf{1}} \otimes \ldots \otimes \mathrm{u}_{k_m}\right) = \sum_{\vec{\boldsymbol{k}} \in \mathcal{A}_{(\mathbf{1},m),N}} \alpha(\vec{\boldsymbol{k}})P_\Omega\left(\mathrm{u}_{k_\mathbf{1}}, \ldots, \mathrm{u}_{k_m}\right).$$

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- Computation of each term of the derivative.

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□

REMARK : the expression of the functional and its derivative are fully deterministic, and depend on the first m stochastic moments of the random variables.

Numerical tools

Mesh generation and adaptation

- **medit** [4] for the mesh generation in 3D ;
- **mmg platform** for the mesh adaptation (in 2D and 3D) ;

Finite-element method

- **FreeFem++** [5] ;

Level-set function

- **mshdist** [6] for the computation of the signed-distance level-set function ;
- **advect** [6] for the solution of the advection equation ;

Optimization algorithm

- **nullspace optimization** developed in python.

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5. Coupled with python using the packages *pymedit* and *pyfreefem* respectively.
6. Available in the ISCD toolbox.

Complexity (1)

We denote $\mathcal{T}(P_\Omega(m), N)$ the number of terms appearing in the expression of the shape derivative of a m-multilinear functional, when the load $\mathrm{g} \in \mathrm{L}^m\left(\mathrm{L}^2(\Gamma_\mathrm{N}); \mathbb{P}\right)$ is decomposed in N mutually independent random variables.

For a generic multilinear functional P_Ω, the number of terms to consider increases exponentially with m :

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The functional $G_\Omega(\cdot, \dots, \cdot)$ related to the m-th norm of the von Mises stress presents some symmetries but is not fully symmetric.

$$\mathcal{T}(G_\Omega(m), N) = \binom{\frac{N(N+1)}{2} + \frac{m}{2} - 1}{\frac{m}{2}}$$

The optimum as an intermediate shape

Heigth of the domain	2.5
Length of the domain	3.0
Elastic coefficients	
Young's modulus E	15
Poisson's ration ν	0.35
Mechanical loads	
Horizontal term g_x	1.0
Vertical term g_y	1.0
Stochastic moments	
Expectations $\mathbb{E}[\xi_x] = \mathbb{E}[\xi_y]$	0.0
Variances $\mathrm{Var}[\xi_x] = \mathrm{Var}[\xi_y]$	1.0
Correlation $\mathbb{E}[\xi_x \, \xi_y]$	0.0

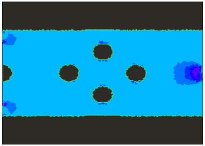


Figure – Initial condition for the three simulations

Figure – Optimized shape for a **vertical** load

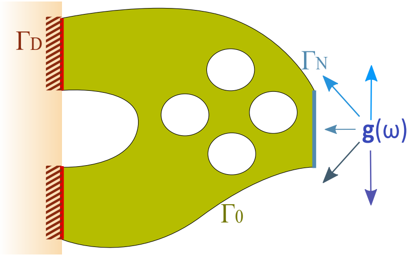


Figure – Illustration of the problem : $g(\omega) = g_x \xi_x(\omega) + g_y \xi_y(\omega)$.

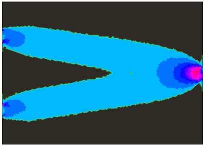


Figure – Optimized shape for a **horizontal** load

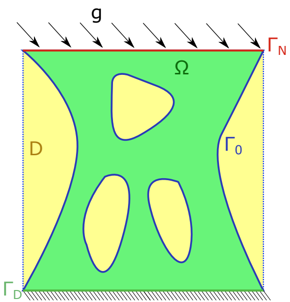
Figure – Optimized shape for an **uncertain** load

Stochastic moments of the compliance

We consider a 2D elastic structure subject to a mechanical load on the upper surface.
The mechanical load g is a **random variable** such that :

$$g(\omega) = g_0 + g_X X(\omega) + g_Y Y(\omega).$$

We would like to minimize the volume of the structure under constraintson the **average** and **variance** of the mechanical compliance.



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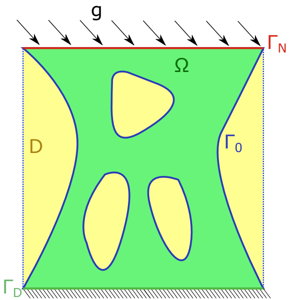

We would like to minimize the volume of the structure under constraintson the **average** and **variance** of the mechanical compliance.

Since the compliance is a **quadratic** functional, its variance can be written as the expectation of a 4 degree polynomial of the state.

Expression of the random variables

We consider the following family of random variables, depending on the real parameters $\alpha \in [0, \frac{\pi}{2}[$ and $\beta \in [0, \frac{\pi}{2}[$:

$$X_\alpha = T \sin(\alpha) + N_X \cos(\alpha); \qquad Y_{\alpha,\beta} = \frac{\sin(\beta)}{\sqrt{\mathrm{Var}[X_\alpha^2]}} \left(X_\alpha^2 - \mathbb{E}\left[X_\alpha^2\right]\right) + N_Y \cos(\beta),$$

where $T \sim \mathcal{U}(\{-1,1\})$, $N_X \sim \mathcal{N}(0,1)$, and $N_Y \sim \mathcal{N}(0,1)$ are independent random variables.

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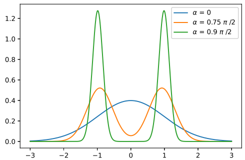


Figure – Density of X_α for different values of α.

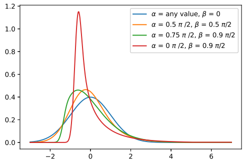


Figure – Density of $Y_{\alpha,\beta}$ for different values of α, β.

For all choice of α and β, the random variables X_α and $Y_{\alpha,\beta}$ are centered, normalized and decorrelated, but not necessarily independent.

$$\mathbb{E}[X_\alpha] = \mathbb{E}\left[Y_{\alpha,\beta}\right] = 0; \qquad \mathbb{E}\left[X_\alpha^2\right] = \mathbb{E}\left[Y_{\alpha,\beta}^2\right] = 1; \qquad \mathbb{E}\left[X_\alpha Y_{\alpha,\beta}\right] = 0.$$

2D optimization problem

We consider a set of 2D admissible shapes $\mathcal{O}_{adm}$, sharing the portions Γ_D and Γ_N, a space of events $\mathcal{O}$, and a probability measure $\mathbb{P}$.

We suppose that the random mechanical load g is applied to Γ_N.

Find $\Omega \in \mathcal{O}_{adm}$

minimizing $\Omega \mapsto \mathrm{Vol}(\Omega)$,

such that, for all $\omega \in \mathcal{O}$,

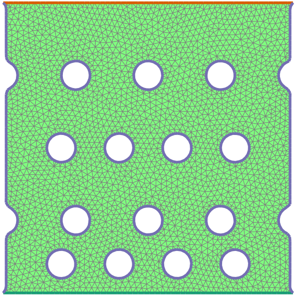
the state $u_{\Omega,g}(\omega) \in \left[\mathrm{H}^1(\Omega)\right]^d$ solves :

$$\begin{cases} -\mathrm{div}\,\boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) = 0 & \text{in } \Omega, \\ \boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) n = g(\omega) & \text{on } \Gamma_N, \\ \boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) n = 0 & \text{on } \Gamma_0, \\ u_{\Omega,g}(\omega) = 0 & \text{on } \Gamma_D. \end{cases}$$

and the following constraints hold :

$$\mathbb{E}\left[C_2(\Omega, u_{\Omega,g})\right] \leq M_0,$$

$$\mathrm{Var}\left[C_2(\Omega, u_{\Omega,g})\right] \leq M_1,$$



where

$$C_2(\Omega, u_{\Omega,g}) = \int_\Omega \boldsymbol{\sigma}\left(u_{\Omega,g}(\omega)\right) : \boldsymbol{\varepsilon}\left(u_{\Omega,g}(\omega)\right) \mathrm{dx}.$$

Numerical parameters and results (1)

Heigth of the domain	1.0
Length of the domain	1.0
Mesh size parameters	
minimal element size `hmin`	0.01
maximal element size `hmax`	0.02
gradation value `hgrad`	0.5
Elastic coefficients	
Young's modulus E	15
Poisson's ration ν	0.35
Mechanical loads	
Fixed load g_0	1.2
Horizontal term g_x	1.0
Vertical term g_y	0.3
Tresholds for the inequality constraints	
Treshold for the expected value M_0	2.0
Treshold for the variance M_0	3.0625
Number of iterations	500

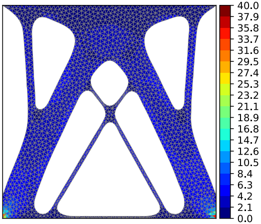


Figure – Case 1 : $\alpha = 0.0$, $\beta = 0.95$

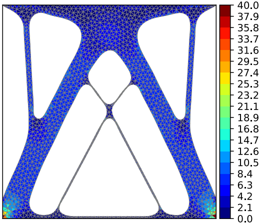
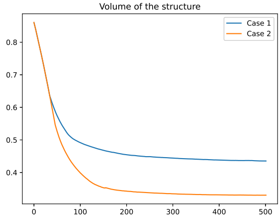
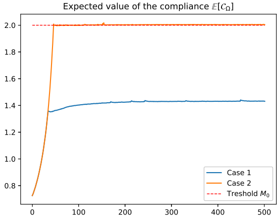


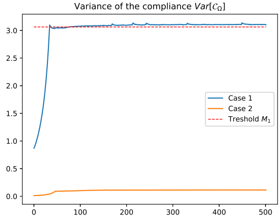
Figure – Case 2 : $\alpha = 0.95$, $\beta = 0.0$

Numerical parameters and results (2)

	Case 1	Case 2
α β	0.0 $0.95\ \frac{\pi}{2}$	$0.95\ \frac{\pi}{2}$ 0.0
Duration (minutes)	25.82	30.80
Final volume $\mathrm{Vol}(\Omega)$	0.435348	0.330348
Saturation of the constraints $\mathbb{E}\left[C_2(\Omega, u_{\Omega,g})\right] - M_0$ $\mathrm{Var}\left[C_2(\Omega, u_{\Omega,g})\right] - M_1$	 −0.56886 0.03952	 0.004041 −2.94801







The expected value as a constraint

In the shape optimization problem we have considered the expectation of the m-th power of the L^m-norm of the von Mises stress s_{VM} as a constraint.

An upper bound on $\mathbb{E}\left[\|s_{\mathrm{VM}}\|^m_{L^m(\Omega)}\right]$ allows to bound $\mathbb{E}\left[\|s_{\mathrm{VM}}\|_{L^m(\Omega)}\right]$ as well thanks to the convexity of $x \mapsto |x|^m$:

$$\mathbb{E}\left[\|s_{\mathrm{VM}}\|^m_{L^m(\Omega)}\right] \leq \mathbb{E}\left[\|s_{\mathrm{VM}}\|_{L^m(\Omega)}\right]^m \leq M_0^m.$$

However, there is no reason to assume that an admissible shape $\Omega \in \mathcal{O}_{adm}$ minimizing $\mathbb{E}\left[\|s_{\mathrm{VM}}\|^m_{L^m(\Omega)}\right]$ is also a minimizer for $\mathbb{E}\left[\|s_{\mathrm{VM}}\|_{L^m(\Omega)}\right]$.