# Numerical computation of the fractional Laplacian with $\alpha = 1$ by means of Gaussian hypergeometric functions and fast convolution methods*

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In this paper, we develop a fast and spectrally accurate pseudospectral method to approximate numerically the fractional Laplacian $(-\Delta)^{1/2}$ on $\mathbb{R}$. More precisely, the change of variable $x = L\cot(s)$, with $L > 0$ and $s \in (0, \pi)$, we allow us to map $\mathbb{R}$ into a finite interval where spectral or psedo-spectral methods might be performed. Indeed, we have obtained the following representation for the fractional Laplacian with $\alpha = 1$ in this new variable applied to elementary trigonometric functions :

$$(-\Delta)^{1/2}e^{ik\operatorname{arccot}(x/L)} = (-\Delta)_s^{1/2}e^{iks} = \frac{-2i}{L\pi(k+2)} - \frac{k}{L}\sin^2(s)e^{iks} + \frac{8i\ {}_2F_1\left(1, -k/2-1; -k/2+2; e^{i2s}\right)}{L\pi(4-k^2)}$$
$$= \frac{-2i}{L\pi(k+2)} - \frac{2ik}{L\pi}e^{iks}\left(\cos(s) + \sin^2(s)\ln\left(\cot\left(s/2\right)\right) + \sum_{n=0}^{(k-1)/2}\frac{4e^{-i(2n+1)s}}{(2n-1)(2n+1)(2n+3)}\right)$$

for an odd positive number $k$. Using the fact that $(-\Delta)_s^{1/2}e^{iks} = \overline{(-\Delta)_s^{1/2}e^{-iks}}$, we obtain it for odd negative numbers. Moreover, the other crucial point of this paper, is the use of a fast convolution algorithm to compute simultaneously $(-\Delta)^{1/2}e^{iks}$ for extremely large sets of odd values of $n$ :

**Lemme 1.** *Let $M \in \mathbb{N}$, $a_0, a_1, \ldots, a_M \in \mathbb{C}$, and $s \in (0, \pi)$. Then,*

$$\sum_{l=0}^{M} a_l(-\Delta)_s^{1/2}e^{i(2l+1)s} = \frac{-i}{L\pi}\sum_{l=0}^{M}\frac{2a_l}{2l+3} - \frac{i}{L\pi}\Big(\cos(s) + \sin^2(s)\ln\left(\cot\left(s/2\right)\right)\Big)\sum_{l=0}^{M}(4l+2)a_l e^{i(2l+1)s} + \frac{i}{L\pi}\sum_{l=0}^{M}(\tilde{b}*\tilde{c})_l e^{i2ls}, \tag{1}$$

*where $P$ can be any natural number such that $P \geq 2M+1$, and $(\tilde{b}*\tilde{c})_l$ denotes the discrete convolution of the $P$-periodic sequences $\tilde{b}_l$ and $\tilde{c}_l$, where*

$$\tilde{b}_l = \begin{cases} (16l+8)a_l, & 0 \leq l \leq M, \\ 0, & M+1 \leq l \leq P-1, \end{cases}$$

*and*

$$\tilde{c}_l = \begin{cases} 1/3, & l = 0, \\ 0, & 1 \leq l \leq P-M-1, \\ \dfrac{1}{(2(l-P)-3)(2(l-P)-1)(2(l-P)+1)}, & P-M \leq l \leq P-1. \end{cases} \tag{2}$$

The fact that $(\tilde{b}*\tilde{c})_l$ in (1) allows us to compute by means of a fast convolution in $\mathbf{O}(P\log(P))$ operations, whereas if we do a direct calculation we need $\mathbf{O}(P^2)$ operations. Because of this, we are capable of developing an efficient pseudospectral method for the computation of $(-\Delta)_s^{1/2}u(x)$.

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