Vlasov-Maxwell equations in a plasma, cold plasma model, the Ordinary and the eXtraordinary mode

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SMAI 2023, Le Gosier, Guadeloupe, 25 mai 2023

Outline

1. Crash introduction to fusion and magnetically confined plasmas.
2. From Vlasov equation to the cold plasma model
3. The ordinary mode: it is not a singularity
4. Energy deposit at the eXtraordinary point: a regular singular point

Energy from nuclear fusion

- **Fusion research**: Exploit nuclear fusion reactions as a sustainable and clean energy source.
- Fusion reaction requires
  - density n (number of nuclei per unit volume),
  - temperature T (kinetic energy of the nuclei),
  - energy confinement time τ_E (energy / power losses).
- At high temperatures, matter is ionized: **plasma**.
- Plasma confinement:

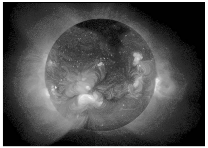
Target reaction:

$$D + T \rightarrow He(3.5MeV) + n(14.1MeV).$$

Stars (p-p cycle):

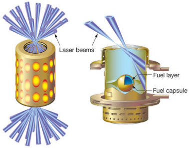
$$H + H \rightarrow D + e^+ + \nu_e,$$
$$D + H \rightarrow {}^3He + \gamma,$$
$${}^3He + {}^3He \rightarrow He + 2H.$$

gravitational



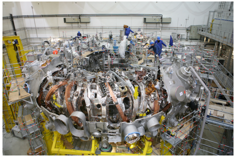
(M. Aschwanden, Physics of the Solar Corona, Praxis Pub. 2005)

inertial



(https://lasers.llnl.gov/science/icf)

magnetic



(Max Planck IPP)

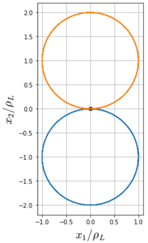
Magnetically confined plasmas

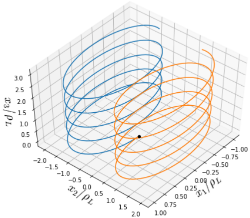
Le livre de la Nature est écrit en langage mathématique (Galiléo Galilée)

- Charged particles in a magnetic field,

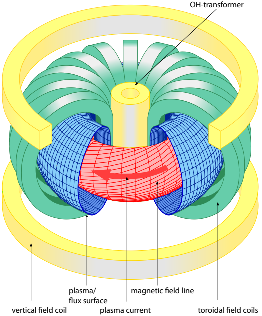
$$\begin{cases} \dfrac{dx}{dt} = v, & \omega_c(x) = \dfrac{|qB(x)|}{mc}, \\ \dfrac{dv}{dt} = \dfrac{q}{mc} v \times B(x), & \rho_L = \dfrac{v_\perp}{\omega_c(x)}. \end{cases}$$

- Uniform field: $B = (0, 0, B_3)$, $B_3 > 0$,

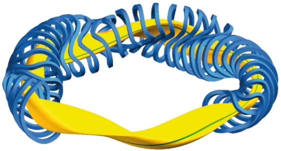




- Particle confinement ≈ field-line confinement.



tokamak

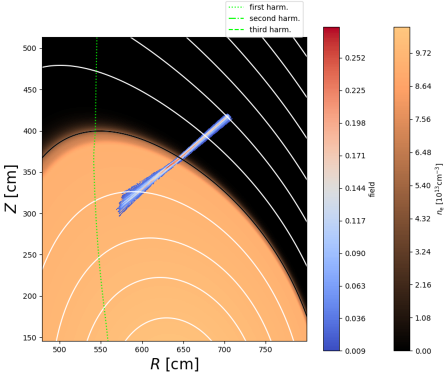


stellarator

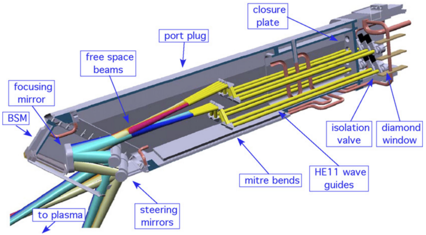
Wave beams for plasma heating and control

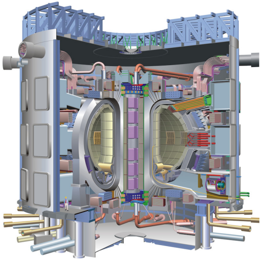
Wave-particle resonances:

$$E(t,x) = E_0 \cos(k \cdot x - \omega t),$$
$$\omega - k_{\parallel} v_{\parallel} - n\omega_c(x) = 0, \quad n \in \mathbb{Z}.$$



M.A. Henderson et al. / Fusion Engineering and Design 82 (2007) 454–462





ITER

System of equations (general)

$$\begin{cases} \partial_t \mathsf{f}_\mathsf{s} + v_\mathsf{s} \cdot \nabla_x \mathsf{f}_\mathsf{s} + q_\mathsf{s}\big(\mathsf{E} + v_\mathsf{s} \times \mathsf{B}/c\big) \cdot \nabla_p \mathsf{f}_\mathsf{s} = C_\mathsf{s}(\{\mathsf{f}_{\mathsf{s}'}\}_{\mathsf{s}'}), \\ \partial_t \mathsf{E} - c\nabla \times \mathsf{B} = -4\pi \sum_\mathsf{s} q_\mathsf{s} \int_{\mathbb{R}^3} v_\mathsf{s}(p)\mathsf{f}_\mathsf{s}(t,x,p)dp, \\ \partial_t \mathsf{B} = -c\nabla \times \mathsf{E}, \\ \nabla \cdot \mathsf{B} = 0, \\ \nabla \cdot \mathsf{E} = 4\pi \sum_\mathsf{s} q_\mathsf{s} \int_{\mathbb{R}^3} \mathsf{f}_\mathsf{s}(t,x,p)dp, \end{cases} \tag{1}$$

relativistic velocity: $v_\mathsf{s} = p/[m_\mathsf{s}\gamma_\mathsf{s}(p)]$, with $\gamma_\mathsf{s}(p) = \big(1 + p^2/m_\mathsf{s}^2 c^2\big)^{1/2}$,
C_s : relativistic Landau collision operator
c: speed of light, m_s: mass of particles of the species s, q_s: electric charge.

Externally imposed electric field E, given magnetic field B_0, of perturbation B.
System of partial differential equations for (f_s, B) which reads

$$\begin{cases} \partial_t f_\mathsf{s} + v_\mathsf{s} \cdot \nabla_x f_\mathsf{s} + q_\mathsf{s}(v_\mathsf{s} \times B_0/c) \cdot \nabla_p f_\mathsf{s} = -q_\mathsf{s}\big(E + v_\mathsf{s} \times B/c\big) \cdot \nabla_p F_{\mathsf{s},0}, \\ \partial_t B = -c\nabla \times E. \end{cases}$$

or, introducing $g_\mathsf{s} = \partial_t f_\mathsf{s}$

$$\begin{cases} \partial_t g_\mathsf{s} + v_\mathsf{s} \cdot \nabla_x g_\mathsf{s} + q_\mathsf{s}(v_\mathsf{s} \times B_0/c) \cdot \nabla_p g_\mathsf{s} = -q_\mathsf{s}\big(\partial_t E - v_\mathsf{s} \times \nabla \times E\big) \cdot \nabla_p F_{\mathsf{s},0}, \\ \partial_t B = -c\nabla \times E. \end{cases} \tag{2}$$

The fluid model

Assume $B_0 = B_0(x_1)e_3$. Maxwellian $F_0(p) = e^{-\frac{p^2}{2k_BT}}$, $p = mv$.

$$\partial_t f + (v_1\partial_1 f + v_2\partial_2 f + v_3\partial_3 f) + \nu f + eB_0(x_1)(v_2\partial_{p_1} - v_1\partial_{p_2})f = -e(S_1\partial_{p_1}F_0 + S_2\partial_{p_2}F_0 + S_3\partial_{p_3}F_0).$$

$$\partial_t(v_1 f) + v_1(v.\nabla_x f) + \nu v_1 f + eB_0(x_1)(\partial_{p_1}(v_1v_2f) - \frac{v_2}{m}f - \partial_{p_2}(v_1^2 f)) = -e[S.\nabla_p(v_1F_0) - \frac{1}{m}S_1F_0].$$

With $j_l = en_0(x_1)\int_{\mathbb{R}^3} v_l f(t,x,v)dv$, one gets

$$\begin{aligned}\partial_t j_1(t,x_1) + \int_{\mathbb{R}^3} v_1(v.\nabla_x f)dv + \nu j_1(t,x) - \tfrac{e^2B_0(x_1)}{m}j_2(t,x_1) \quad &= -e[n_0(x_1)\int_{\mathbb{R}^3} S.\nabla_p(v_1F_0)dv] \\ &+ \tfrac{en_0(x_1)}{m}\int_{\mathbb{R}^3} S_1(t,x,v)F_0(p)dv].\end{aligned}$$

Cold plasma model: strong hypothesis $f \simeq e^{-\frac{p^2}{2k_BT}}$, $< v >= 0$, $< v^2 > \simeq k_BT \simeq 0$.

$$\Rightarrow \partial_t j_1(t,x_1) + \nu j_1(t,x_1) - \frac{eB_0(x_1)}{m}j_2(t,x_1) = \frac{e^2n_0(x_1)}{m\varepsilon_0}E_1(t,x).$$

One deduces in a similar way

$$\begin{aligned}\partial_t j_2(t,x_1) + \nu j_2(t,x_1) + \tfrac{eB_0(x_1)}{m}j_1(t,x_1) \quad &= \tfrac{e^2n_0(x_1)}{m}E_2(t,x), \\ \partial_t j_3(t,x_1) + \nu j_3(t,x_1) \quad &= \tfrac{e^2n_0(x_1)}{m\varepsilon_0}E_3(t,x).\end{aligned}$$

Ordinary mode (cyclotron frequency)

Let $\omega_c(x_1) = \frac{eB_0(x_1)}{m}$, $\omega_p(x_1) = \sqrt{\frac{e^2 n_0(x_1)}{m\varepsilon_0}}$ (cyclotron frequency and plasma frequency).

$$\begin{cases} m\partial_t \mathbf{v} = e(E + \mathbf{v} \times B_0) \\ \mathbf{j} = en_0(x_1)\mathbf{v} \\ \nabla \times B - c^{-2}\partial_t E = \mu_0 j, \nabla \times E + \partial_t B = 0 \end{cases}$$

Yields $\partial_t \mathbf{j} + \omega_c(x_1) e_3 \times \mathbf{j} + \nu \mathbf{j} = (\omega_p(x_1))^2 \varepsilon_0 \mathbf{E}$.

[1] and for $\omega = 0$

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Exists a unique solution of this Cauchy problem without initial condition, in $\mathcal{S}'(\mathbb{R}_t)$.
After Fourier transform in time:

$$\begin{cases} (i\omega + \nu) j_1 - \omega_c(x_1) j_2 & = (\omega_p(x_1))^2 \varepsilon_0 E_1 \\ (i\omega + \nu) j_2 + \omega_c(x_1) j_1 & = (\omega_p(x_1))^2 \varepsilon_0 E_2 \\ (i\omega + \nu) j_3 & = (\omega_p(x_1))^2 \varepsilon_0 E_3 \end{cases} \Leftrightarrow \begin{cases} j_1 & = \frac{(\omega_p(x_1))^2 \varepsilon_0}{\omega_c^2 + (i\omega+\nu)^2} (\omega_c E_2 + (i\omega + \nu) E_1) \\ j_2 & = \frac{(\omega_p(x_1))^2 \varepsilon_0}{\omega_c^2 + (i\omega+\nu)^2} (-\omega_c E_1 + (i\omega + \nu) E_2) \\ j_3 & = \frac{(\omega_p(x_1))^2}{i\omega+\nu} \varepsilon_0 E_3 \end{cases}$$

Describes the conductivity operator:

$$\mathbf{j} = \sigma(x_1, \omega)\mathbf{E}.$$

Response of the plasma, matrix $\sigma(x_1, \omega)$ is singular when $\nu \to 0_+$ for $\omega = \omega_c(x_1)$[1].

[1] and for $\omega = 0$

Apparent singularity:

Proposition

At a point x_1^c such that $\omega = \omega_c(x_1^c)$, the electric field (E_1, E_2) goes to a finite limit when $\nu \to 0_+$. No resonant heating at the cyclotron frequency.

Proof: it will come as a byproduct of the global system on $\mathbf{E}, \mathbf{B}, \mathbf{j}$.

The resonant heating is obtained as a consequence of

Theorem

Let x_1^h be the unique solution of $\omega = \sqrt{\omega_p^2(x_1) + \omega_c^2(x_1)}$, assume $\partial_{x_1}(\omega_p^2 + \omega_c^2)(x_1^h) \neq 0$.
There is a resonant heating at x_1^h, that is $\int_B (\mathbf{E}.\mathbf{B}) dx_1 \to iQ_0$ when $B \to \{x_1^h\}$.
It is a consequence of $E_1 \simeq \frac{1}{x_1 - x_1^h}$.

Proof: After Fourier transform in time and Fourier transform in (x_2, x_3), replacing x_1 by x ad using $\omega_\nu = \omega - i\nu$

$$\begin{cases} ik_3 E_1 - \partial_x E_3 = i\omega\mu_0 H_2 \\ \partial_x E_2 - ik_2 E_1 = i\omega\mu_0 H_3 \\ ik_3 H_1 - \partial_x H_3 = j_2 - i\omega\varepsilon_0 E_2 \\ \partial_x H_2 - ik_2 H_1 = j_3 - i\omega\varepsilon_0 E_3 \\ ik_2 E_3 - ik_3 E_2 = i\omega\mu_0 H_1 \\ -i\omega_\nu j_3 = \varepsilon_0 \omega_p^2 E_3 \\ j_1 - i\omega\varepsilon_0 E_1 = ik_2 H_3 - ik_3 H_2 \\ -i\omega_\nu j_1 + \omega_c j_2 = \varepsilon_0 \omega_p^2 E_1 \\ -\omega_c j_1 - i\omega_\nu j_2 = \varepsilon_0 \omega_p^2 E_2. \end{cases} \tag{3}$$

The subsystem of equations, in the plane (x_1, x_2) without derivatives is

$$\left\{ \begin{array}{llll} j_1 & & -i\omega\varepsilon_0 E_1 & = ik_2 H_3 - ik_3 H_2 \\ -i\omega_\nu j_1 & +\omega_c j_2 & -\varepsilon_0 \omega_p^2 E_1 & = 0 \\ -\omega_c j_1 & -i\omega_\nu j_2 & & = \varepsilon_0 \omega_p^2 E_2 \end{array} \right. \tag{4}$$

with, in addition

$$H_1 = \frac{ik_2 E_3 - ik_3 E_2}{i\omega\mu_0}, j_3 = -\frac{ik_2 H_3 - ik_3 H_2}{i\omega_\nu}.$$

Its determinant is $D_\nu(x) = -i\omega_\nu \omega_p^2(x) - i\omega(\omega_c^2(x) - \omega_\nu^2)$, with

$$D_0(x) = -i\omega[\omega_p^2(x) + \omega_c^2(x) - \omega^2]. \tag{5}$$

Resulting system

$$D_\nu(x)\frac{d}{dx}U = M_\nu(x, k_2, k_3)U, \quad U = (E_3, H_2, E_2, H_3)^T.$$

As D_0 does not vanish at $x = x_1^h$, no singularity of the system. A turning point appears and this does not lead to a singular solution.

Normal incidence $k_3 = 0$. The system becomes

$$\begin{cases} -\partial_x E_3 = i\omega\mu_0 H_2 \\ \partial_x E_2 - ik_2 E_1 = i\omega\mu_0 H_3 \\ -\partial_x H_3 = j_2 - i\omega\varepsilon_0 E_2 \\ \partial_x H_2 - ik_2 H_1 = j_3 - i\omega\varepsilon_0 E_3, \end{cases}$$

with $H_1 = \frac{k_2 E_3}{\omega\mu_0}, j_3 = -\frac{k_2 H_3}{\omega_\nu}$, and j_1, j_2, E_1 expressed with H_3, E_2. Deduce the system on (E_2, H_3):

$$\begin{cases} \partial_x E_2 - ik_2 E_1 = i\omega\mu_0 H_3 \\ -\partial_x H_3 = j_2 - i\omega\varepsilon_0 E_2, \end{cases}$$

and the system on (E_3, H_2) with H_3 as source term.
There exists $A_\nu(x_1), B_\nu(x_1), C_\nu(x_1)$, such that $A_\nu(x_1)D_\nu(x_1), B_\nu(x_1)D_\nu(x_1), C_{\nu(x_1)}D_\nu(x_1)$ smooth and non zero in a neighborhood of x_1^h (that is, for example, $A_\nu \simeq \frac{a}{x-x_1^h}$) and

$$\begin{cases} \varepsilon_0 E_1 = A_\nu(x_1)\varepsilon_0 E_2 + B_\nu(x_1) ik_2 H_3, \\ j_2 = -C_\nu(x_1)\varepsilon_0 E_2 + A_\nu(x_1) ik_2 H_3 \end{cases}$$

Deduce

$$[\frac{H_3'}{i\omega + C_\nu}]' + (\frac{A_\nu}{i\omega + C_\nu})' H_3 = (i\omega\mu_0 - k_2^2 \frac{A_\nu^2 + B_\nu C_\nu + i\omega B_\nu}{i\omega + C_\nu}) H_3. \quad (6)$$

Treating the singularity in the ODE

Physics: only known a particular solution in the case $(i\omega + C_0)^{-1} = 1 - \frac{x}{x_1^h}$: called the Budden problem.

Properties used here to address completely the general case:

1. Analyticity and there exists X_ν such that $D_\nu(X_\nu) = 0$ and $X_0 = x_1^h$. (property of the plasma)
2. $D_\nu(x)(A_\nu^2 + B_\nu C_\nu)$ is bounded for (x, ν) in a neighborhood $\mathcal{V}$ of $(x_1^h, 0)$ in $\mathbb{C}^2$. (result of the calculus)

New unknown $W = \frac{H_3}{(i\omega + C_\nu)^{\frac{1}{2}}}$. ODE on W

$$W'' = \quad (i\omega\mu_0 - k_2^2 \frac{A_\nu^2 + B_\nu C_\nu + i\omega B_\nu}{i\omega + C_\nu})(i\omega + C_\nu)W + \quad [(i\omega + C_\nu)^{-\frac{1}{2}}]''(i\omega + C_\nu)^{\frac{1}{2}} W$$

$$(x - X_\nu)^{-1} \qquad\qquad\qquad\qquad (x - X_\nu)^{-2}.$$

Equation for the general problem

$$W'' = (\frac{R_\nu(x)}{x - X_\nu} - \frac{1}{4(x - X_\nu)^2})W. \tag{7}$$

Model problem

Equation with frozen coefficients

$$\frac{d^2Y}{dx^2} = (\frac{R_\nu(X_\nu)}{x - X_\nu} - \frac{1}{4(x - X_\nu)^2})Y.$$

Scaling $x - X_\nu = \kappa z$;

Model problem

Equation with frozen coefficients

$$\frac{d^2Y}{dx^2} = (\frac{R_\nu(X_\nu)}{x - X_\nu} - \frac{1}{4(x - X_\nu)^2})Y.$$

Scaling $x - X_\nu = \kappa z$;

$$\frac{d^2Y}{dz^2} = (\frac{\kappa R_\nu(X_\nu)}{z} - \frac{1}{4z^2})Y.$$

Choice $\kappa R_\nu(X_\nu) = -\frac{1}{2}$. Equation becomes

$$\frac{d^2Y}{dz^2} = (-\frac{1}{2z} - \frac{1}{4z^2})Y,$$

which solutions are

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which solutions are

$$\begin{aligned} Y(z) &= \sqrt{z}(AJ_0(\sqrt{z}) + BY_0(\sqrt{z})) \\ &= \sqrt{z}(AJ_0(\sqrt{z}) + B[Y_0(\sqrt{z}) - \tfrac{2}{\pi}\ln\tfrac{z}{2}J_0(\sqrt{z})] + \tfrac{2B}{\pi}\ln\tfrac{z}{2}J_0(\sqrt{z})). \end{aligned}$$

Solution of (7) (the general problem)

Construct an exact pair of solutions of the equation $f'' = [\frac{R_\nu(X_\nu)}{\rho(x)} - \frac{1}{4(\rho(x))^2} + s(x)]f$.
Want to use them for constructing solutions of $W'' = [\frac{R_\nu(x)}{x-X_\nu} - \frac{1}{4(x-X_\nu)^2}]W$.
Duhamel's principle (variation of the constants).
Singular solution:

$$Y \simeq \sqrt{\rho(x)}Y_0(\sqrt{\rho(x)}) \simeq \frac{2}{\pi}\sqrt{\rho(x)}\ln\sqrt{\rho(x)}.$$

As E_1 is the derivative of Y, one gets the singularity of the solution of Maxwell's equation near a plasma, hence energy deposit across this singularity.

Stretching function $\rho(x)$. One has

$$\frac{d}{dx}(Y(\rho(x)) = \rho'(x)Y'(\rho(x)) \Rightarrow \frac{d}{dx}(\frac{\frac{d}{dx}(Y(\rho(x)))}{\rho'(x)}) = \rho' Y''(\rho) = \rho'(-\frac{1}{2\rho} - \frac{1}{4\rho^2})Y(\rho)$$

Equation for the stretching function

$$(\rho')^2(\frac{R_\nu(X_\nu)}{\rho} - \frac{1}{4\rho^2}) = \frac{R_\nu(X_\nu)}{x - X_\nu} - \frac{1}{4(x-X_\nu)^2} \Leftrightarrow \frac{\rho'}{\rho} = \frac{1}{x - X_\nu}\sqrt{\frac{1-4(x-X_\nu)R_\nu(x)}{1-4\rho R_\nu(X_\nu)}} \quad (8)$$

Calculation of the stretching function: write $\rho(x) = (x - X_\nu)\tau(x) \Rightarrow$

$$\frac{\tau'}{\tau} = F(x - X_\nu, \tau)$$

with F smooth in a neighborhood of X_ν. Existence and uniqueness of the solution such that $\tau(X_\nu) = 1$ (on a line in the complex plane).

Define

$$f(x) = (\rho'(x))^{-\frac{1}{2}} Y(\rho(x)). \quad (9)$$

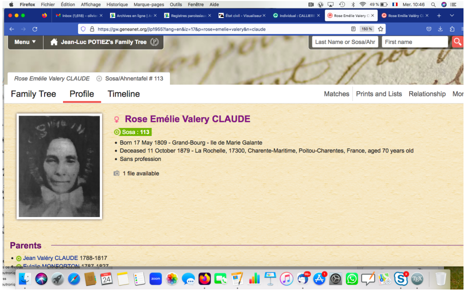
Equation on f:

$$f'' = [\frac{R_\nu(X_\nu)}{x - X_\nu} - \frac{1}{4(x-X_\nu)^2} + s(x)]f, \quad s(x) = \sqrt{\rho'(x)}\frac{d^2}{dx^2}(\frac{1}{\sqrt{\rho'(x)}})$$

- In the study of the plasma heating, passing through the conductivity operator might not be the most efficient method, the complete coupled system is better
- In this type of model, the transient solutions are not considered, because one uses a Fourier transform
- The method of stretching function ($\simeq$ 1950) amounts to solving (in wave propagation) the eikonal equation and use the solution as a new variable
- The general set-up for the conductivity operator uses the 'response' of the plasma as well.
- Known: cold plasma model with oblique incidence in the non-uniform case, and singular behavior of the general conductivity operator in the uniform case.

Thanks for the attention and again thanks to Pascal and SMAI for bridging the Atlantic ocean!

Genealogy-Marie-Galante



(https://lasers.llnl.gov/science/icf)