Solutions classiques pour des équations de type Vlasov-Poisson en domaine borné

joint work with M. Iacobelli

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The classical Vlasov-Poisson

The Vlasov-Poisson system is a non-collisional kinetic equation which models the evolution of a distribution $f(t, x, v)$ of electrons in a plasma :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 & \text{in } (0, +\infty) \times \Omega \times \mathbb{R}^d, \\ E = -\nabla U, \quad \Delta U = -\rho & \text{in } (0, +\infty) \times \Omega, \\ f|_{t=0} = f_0 & \text{in } \Omega \times \mathbb{R}^d \end{cases}$$

with $\rho(t, x) = \int_{\mathbb{R}^d} f \, \mathrm{d}v$ the macroscopic density and $\Omega \subseteq \mathbb{R}^d$.

The ionic Vlasov-Poisson model

If one wishes, instead, to model the evolution of ions in a plasma, then one may use the ionic Vlasov-Poisson, also called Vlasov-Poisson for massless electrons. This model derives from a coupled system. Let us write f_- the distribution of electrons and f_+ the distribution of ions and consider the following classical model for plasma dynamics:

$$\begin{cases} \partial_t f_- + v \cdot \nabla_x f_- + \nabla_x \phi \cdot \nabla_v f_- = Q(f_-) \\ \partial_t f_+ + v \cdot \nabla_x f_+ + \nabla_x \phi \cdot \nabla_v f_+ = 0 \\ \Delta \phi = \rho_- - \rho_+ \end{cases}$$

with $\rho_\pm = \int f_\pm \, \mathrm{d}v$ and $E = \nabla_x \phi$ is the total electric field generated by both electrons and ions.

The ionic Vlasov-Poisson model

The characteristic time-scale of the electrons Vlasov equation is significantly shorter than that of the ions equations.
The *Massless electron* limit describes the asymptotic behaviour of the system as the ratio of mass electron/ions grows small.
Formally, it comes down to assuming that, in the time-scale of the ions, the electrons immediately reach the thermodynamical equilibrium of their collisional kinetic equation, hence

$$f_- \to Ce^{\phi(x)}e^{-|v|^2/2}$$

and the Poisson equation for the electric potential then becomes

$$\Delta\phi = \tilde{C}e^{\phi(x)} - \rho_+.$$

The ionic Vlasov-Poisson model

Writing f the distribution of ions, we get the Vlasov-Poisson for Massless Electrons model:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 & \text{in } (0, +\infty) \times \Omega \times \mathbb{R}^d, \\ E = -\nabla U, \quad \Delta U = e^U - \rho - 1 & \text{in } (0, +\infty) \times \Omega, \\ f|_{t=0} = f_0 & \text{in } \Omega \times \mathbb{R}^d \end{cases}$$

still with $\rho(t, x) = \int_{\mathbb{R}^d} f \, \mathrm{d}v$ and $\Omega \subseteq \mathbb{R}^d$.
Note that we have added a repulsive force, in the form of a "-1" in the Poison equation, in order to ensure a good behaviour of the field in the neighbourhood of $\partial\Omega$. Moreover, this "-1" ensures that if $\rho \equiv 0$ then $U \equiv 0$ is solution to the Poisson equation.

Boundary conditions

Ω is a bounded strictly convex $C^{2,1}$ domain. On the boundary we consider the conditions :

- For the Vlasov equation: specular reflections

$$\gamma_- f(t,x,v) = \gamma_+ f(t,x,v-2(v\cdot n_x)n_x) \quad x \in \partial\Omega, v\cdot n_x < 0$$

- For the Poisson equation, either homogeneous Dirichlet:

$$U(t,x) = 0 \qquad x \in \partial\Omega$$

  or Neumann conditions:

$$\nabla U \cdot n_x = h \qquad x \in \partial\Omega$$

  with $h \in C^{1,\mu}(\partial\Omega)$ satisfying $\int_{\partial\Omega} h \,\mathrm{d}\sigma = -1$ for VP
  and $h < 0$, $\int_{\partial\Omega} |h| \,\mathrm{d}\sigma < 1 + |\Omega|$ for VPME.

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Existence and uniqueness results

Vlasov-Poisson in bounded domains in $\mathbb{R}^3$, assuming the initial condition is compactly supported and constant in a neighbourhood of the grazing set:

- Half-space : Guo '94, propagation of moments in the spirit of Lions, Perthame (91)
- Half-space : Hwang, Velázquez '09, Pfaffelmoser method
- Convex domain : Hwang, Velázquez '10, local flattening of the C^5 boundary of Ω.

Weak solutions in bounded domains: Alexandre '93, Ben Abdallah '94, Weckler '95, Mischler '99, Fernández-Real '18.

Existence and uniqueness results

Vlasov-Poisson for Massless Electrons :

- $\mathbb{R}^3$: Bouchut '91, weak solutions for $f_0 \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \cap \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3)$.
- $\mathbb{R}$: Han-Kwan, Iacobelli '17, weak solutions for $f_0 \in \mathcal{P}_1(\mathbb{T} \times \mathbb{R})$.
- $\mathbb{T}^3$ et $\mathbb{R}^3$: Griffin-Pickering, Iacobelli '21, '21, strong solutions for $f_0 \in \mathcal{P}_1(\mathbb{T} \times \mathbb{R})$ and uniqueness à la Loeper assuming bounded density.

Assumption on f_0

For our existence result, we make the following assumptions on f_0

- $\iint_{\Omega\times\mathbb{R}^3} f_0 \,\mathrm{d}x\,\mathrm{d}v = 1,\ f_0 \geq 0,$
- $f_0 \in C^{1,\mu}(\bar{\Omega}\times\mathbb{R}^3),\ \mu \in (0,1),$
- $\operatorname{supp} f_0 \subset\subset \bar{\Omega}\times\mathbb{R}^3$
- $f_0(x,v) =$ constante on a neighbourhood of γ_0

with γ_0 the grazing set in phase-space:

$$\gamma_0 = \{(x,v) \in \partial\Omega\times\mathbb{R}^3 : v\cdot n(x) = 0\}.$$

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Statement of our results

Theorem (C., Iacobelli)

Let Ω be a $C^{2,1}$ strictly convex domain and f_0 satisfying the previous assumptions. There exists a unique classical solution

$$f \in C^1_t C^{1,\mu'}((0,\infty) \times \Omega \times \mathbb{R}^3), \quad E \in C^1_t C^{2,\mu'}((0,\infty) \times \Omega \times \mathbb{R}^3)$$

for all $\mu' \in (0,\mu)$, to the Vlasov-Poisson system (classical or ionic) with specular reflections on the boundary for the Vlasov equation, and either homogenous Dirichlet or Neumann boundary condition for the Poisson equation, with the proper compatibility assumption for the latter.

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Decoupled system

For all $n \geq 0$ we consider f_0^n satisfying appropriate assumptions, E^0 the associated electric field and the system of equations for $n \geq 1$:

$$\begin{cases} \partial_t f^n + v \cdot \nabla_x f^n + E^{n-1} \cdot \nabla_v f^n = 0 \\ E^n(t,x) = -\nabla U^n, \quad \Delta U^n = e^{U^n} - \displaystyle\int_{\mathbb{R}^d} f^n \, \mathrm{d}v - 1, \\ f^n(0,x,v) = f_0^{n-1}(x,v) \end{cases}$$

with specular reflections for Vlasov, and either Dirichlet or Neumann for Poisson.

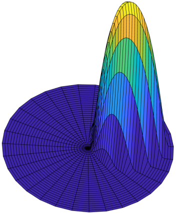
Proposition

Consider $E \in C_t^0 C^{1,\mu}([0,T] \times \bar{\Omega})^d$, $\mu \in (0,1]$ *such that* $E(t,x) \cdot n(x) \geq C_0 > 0$ *for all* $x \in \partial\Omega$. *Then, under the previous assumptions, there exists a unique solution* $f \in C_t^1 C^{1,\mu}([0,T] \times \Omega \times \mathbb{R}^d)$ *to*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 & (t,x,v) \in (0,T] \times \bar{\Omega} \times \mathbb{R}^d, \\ \gamma_- f(t,x,v) = \gamma_+ f(t,x,\mathcal{R}_x v) & (t,x,v) \in (0,T] \times \gamma_-, \\ f|_{t=0} = f_0 & \Omega \times \mathbb{R}^d. \end{cases}$$

Simulations

Figure: Choice of density ρ and associated field E



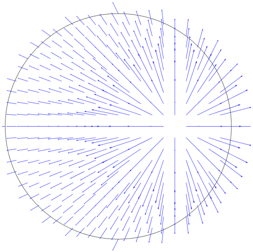


Figure: Trajectory in the disk from $x = (-0.9, 0)$, $v \propto (-0.2, 1)$ and evolution of speed

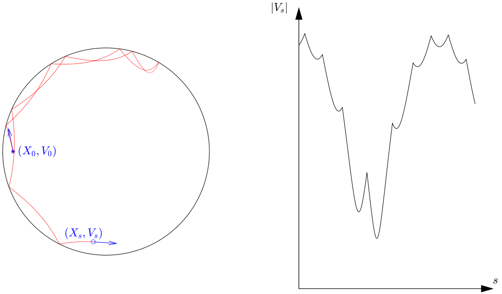
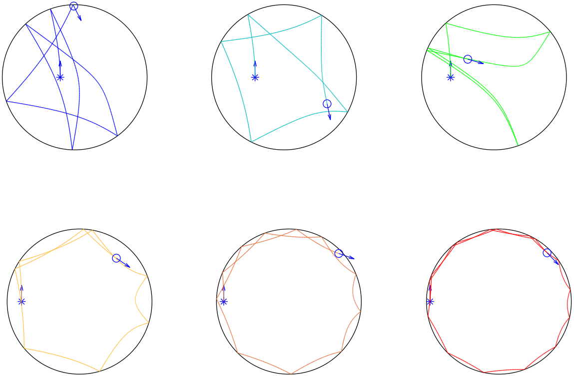


Figure: Examples of trajectories on the disk



Velocity Lemma

In order to construct a classical solution to the Vlasov equation, we construct the flow of transport and propagate the initial condition along that flow. The trajectories of the particles at time s, $(X_s, V_s) = (X(s;t,x,v), V(s;t,x,v))$ is solution to

$$\begin{cases} \partial_s X_s = V_s & X(t;t,x,v) = x, \\ \partial_s V_s = E(s, X_s) & V(t;t,x,v) = v, \\ V_{s^+} = V_{s^-} - 2(n(X_\tau) \cdot V_{s^-}) n(X_s) & \text{for all } s \text{ such that } X_s \in \partial\Omega. \end{cases}$$

Since we assumed $E \in C_t^0 C^{1,\mu}([0,T] \times \bar{\Omega})^d$ the trajectory is well defined in-between reflections on the boundary. Therefore we can construct the full trajectory piece by piece if there are only a finite number of reflections happening in a finite time interval $[s,t]$.

Lemma (Velocity Lemma)

Under the assumptions of the theorem, we define

$$\alpha(t,x,v) = \frac{1}{2}(v\cdot\nabla\xi(x))^2 + \big(v\cdot\nabla^2\xi(x)\cdot v + E(t,x)\cdot\nabla\xi(x)\big)|\xi(x)|.$$

There exists $\delta > 0$ such that α is a $\delta-$kinetic distance.
The grazing set is isolated in the sense that for all (t,x,v) with $dist(x,\partial\Omega) < \delta$

$$C_s^-\alpha(t,x,v) \le \alpha(s, X(s;t,x,v), V(s;t,x,v)) \le C_s^+\alpha(t,x,v)$$

with $C_s^\pm = \exp\big(\pm C_0\big[(|v|+1)|s-t| + \|E\|_{L^\infty}(s-t)^2\big]\big)$,
$C_0 = C_0(\|\xi\|_{C^{2,1}}, \|E\|_{C^{0,1}})$ *when Ω is* ***strictly convex.***

Control of the number of reflections

From the Velocity Lemma, we deduce for all (t, x, v) the number k of reflections of the trajectory $s \to (X(s; t, x, v), V(s; t, x, v))$ on the boundary in the interval of time $s \in (t - \Delta, t)$ is bounded by

$$k \leq \Delta C_1 \frac{(|v| + \Delta \|E\|_{L^\infty})^2 + \|E\|_{L^\infty}}{\sqrt{\alpha(t, x, v)}} e^{C_0[(|v|+1)\Delta + \|E\|_{L^\infty} \Delta^2]}$$

with $C_1 = C_1(\Omega) > 0$.

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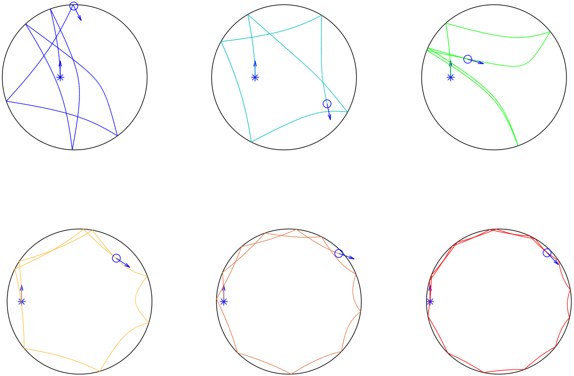
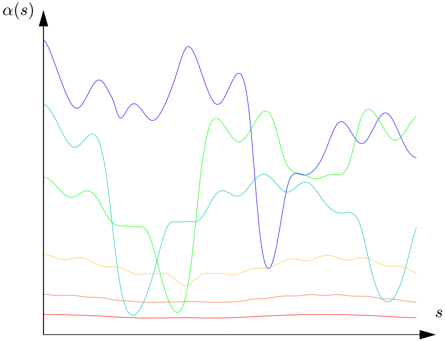
Figure: Examples of trajectories on the disk



Simulations

Figure: Kinetic distances as functions of s



The Dirichlet case

Proposition

Consider $\rho \in C^{0,\alpha}(\Omega)$, $\alpha \in (0,1)$. *The non-linear Poisson equation*

$$\begin{cases} \Delta U = e^U - \rho - 1 & x \in \Omega \\ U(x) = 0 & x \in \partial\Omega. \end{cases}$$

has a unique solution $U \in H_0^1(\Omega)$. *Moreover, this solution is* $C^{2,\alpha}(\Omega)$ *and satisfies* $\partial_n U(x) < 0$ *for all* $x \in \partial\Omega$.

Elements of proof

- Existence and uniqueness in $H_0^1(\Omega)$: we show that there exists a unique minimiser in H_0^1 of the functional

$$\mathcal{E}_D[\phi] := \int_\Omega \left(\frac{1}{2} |\nabla \phi|^2 + e^{\phi} - \phi - \rho\phi \right) \mathrm{d}x$$

- Regularity : we adapt the method of Griffin-Pickering-Iacobelli to the bounded domain case : we decompose U in a regular part $\hat{U}$ and a singular part $\bar{U}$ solutions to

$$\begin{cases} \Delta \hat{U} = e^{\bar{U}+\hat{U}} - 1 \\ \hat{U}|_{\partial\Omega} = 0 \end{cases} \qquad \begin{cases} \Delta \bar{U} = -\rho \\ \bar{U}|_{\partial\Omega} = 0. \end{cases}$$

For $\bar{U}$, classical elliptic regularity : $\bar{U} \in C_c^{2,\alpha}(\bar{\Omega})$.

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Elements of proof

We identify $\hat{U}$ as the unique minimiser in $H_0^1(\Omega)$ of

$$\hat{\mathcal{E}}_D[\phi] := \int_\Omega \left(\frac{1}{2}|\nabla \phi|^2 + e^{\bar{U}+\phi} - \phi \right) \mathrm{d}x.$$

We use the fact that $\bar{U}$ is uniformly bounded i.e. there exists $M_1 > 0$ such that $-M_1 < \bar{U} < M_1$, and the fact that $\hat{U}$ minimises $\hat{\mathcal{E}}_D$ to show that for all $k \in \mathbb{N}$:

$$\|e^{\hat{U}}\|^k_{L^k(\Omega)} \leq C_0 \int_\Omega e^{(k-1)\hat{U}} \,\mathrm{d}x \leq \cdots \leq C_0^{k-1} \|e^{\hat{U}}\|_{L^1(\Omega)} \leq C(\Omega) C_0^k$$

with $C_0 = (1 + e^{M_1})e^{M_1}$. The regularity of $\hat{U}$ and U then follows from classical elliptic regularity and Sobolev embeddings.

Convergence of sequences

Let us introduce

$$Q^n(t) = \sup\{|v| : (x,v) \in \operatorname{supp} f^n(s),\ 0 \leq s \leq t\}$$

and assume there exists $K = K(T)$ such that for all $t \in (0,T)$ et $n \geq 1$: $Q^n(t) \leq K$.

Proposition (Hölder compactness)

Under the assumptions above, the sequences f^n and E^n converge in $C_t^\nu C^{1,\mu'}([0,T] \times \Omega \times \mathbb{R}^d)$ and $C_t^\nu C^{2,\mu'}([0,T] \times \Omega)^d$ for all $\nu < 1$, $\mu' < \mu$ towards f and E. Moreover, these limits are $C_t^1 C^{1,\mu'}([0,T] \times \Omega \times \mathbb{R}^d)$ and $C_t^1 C^{2,\mu'}([0,T] \times \Omega)^d$ respectively and they are solutions to VPME.

Idea of proof: compact velocity support + propagation of L^∞-norm of f $\Rightarrow$ uniform bound on $\rho \Rightarrow$ uniform bound on E. Regularity by iteration.

Bounds on the velocities in 3d

To remove the assumption of bounded velocity support, the key is to control the acceleration along the trajectories of the particles.
Indeed, since we assume supp $f_0 \subset\subset \bar{\Omega} \times \mathbb{R}^d$, if the acceleration is controlled then, at all times, the velocities will be bounded. We show this for the solution (f, E) of the non-linear Vlasov-Poisson system that we obtained above, and we conclude using the convergence of the flow of transport of the approximate problem.
For a given trajectory $(\hat{X}(s), \hat{V}(s))$ we are interested in the quantity

$$\int_{t-\Delta}^{t} |E(s, \hat{X}(s))| \, \mathrm{d}s \leq \int_{t-\Delta}^{t} \iint_{\Omega \times \mathbb{R}^3} \frac{f(s, y, w)}{|y - \hat{X}(s)|^2} \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}s + C\Delta \|h\|_{L^\infty}$$
$$\leq \int_{t-\Delta}^{t} \iint_{\Omega \times \mathbb{R}^3} \frac{f(t, x, v)}{|X(s; t, x, v) - \hat{X}(s)|^2} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s + C\Delta \|h\|_{L^\infty}.$$

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Conclusion

We have thus proved existence of solutions to VPME on $[0, T]$.
We then show that we can take $T \to \infty$ while preserving the bound on the velocities to obtain the global solutions.
Finally, we prove uniqueness by adapting the idea of Lions-Perthame ('91) : consider two solutions f_1 and f_2, they satisfy

$$\partial_t(f^1 - f^2) + v \cdot \nabla_x(f^1 - f^2) + E^1 \cdot \nabla_v(f^1 - f^2) = (E^1 - E^2) \cdot \nabla_v f^2$$

which we interpret as a Vlasov equation, driven by the flow of transport associated to E^1, and we control the right-hand-side by estimations on the Poisson kernel to get

$$\|(f^1 - f^2)(t)\|_{L^1(\Omega\times\mathbb{R}^3)} \leq C(T) \int_0^t \|(f^1 - f^2)(s)\|_{L^1(\Omega\times\mathbb{R}^3)} \, \mathrm{d}s.$$

Thank you for listening !